Characters of groups with normal $*$-subgroups

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INTRODUCTION

The concept of $*$-group was introduced by I.M. Isaacs in [12]. By definition, a group G is a $*$-group if it satisfies the following three conditions: the center $Z(G)$ is cyclic, $G/Z(G)$ is abelian, and every coset of $Z(G)$ in G contains an element x such that $Z(G)\cap\langle x\rangle=\{1\}$.

The best known examples of $*$-groups are the extra special p-groups of exponent p for odd primes p and the central product of an extra special 2-group with a cyclic group of order 4. There are other types of $*$-groups of prime power order. See Example (1.21).

In [12] a host of properties on $*$-groups was derived and the profit of introducing $*$-groups was showed.

The $*$-groups come into the picture whenever character triples are involved, as known from Theorem (1.16), Corollary (1.17), and the remarks made after Lemma (1.19). Character triple theory is an important tool in proving theorems in representation theory of finite groups. Therefore it looks fruitful to extend our knowledge on $*$-groups for future purposes.

It is our objective to present character extension theory and correspondence theory of characters for groups with normal $*$-subgroups (§ 2), and to present some of the structure theory of $*$-groups (§ 1) such as the Theorems (1.7), (1.8) and (1.14). In § 1 some theory of so-called character triples is recorded, useful for $*$-groups and other types of groups. Moreover in § 1 we have derived some character theory of $*$-groups necessary for understanding the results in § 2. Further detailed preliminary information is given in an introduction at § 2.

The main results of this paper are to be found in § 2, notably Theorems (2.4), (2.11), (2.12) and (2.16).

(2.4) THEOREM Let G, H, K, E be finite groups satisfying
(a) E is a normal $*$-subgroup of G of odd order.
(b) H is a complement of E in the semidirect product $G=H\cdot E$.
(c) $K \trianglelefteq H$.
(d) gcd $(|K|, |E|)=1$.
(e) $[E, K]=E$.
(f) $Z(E) \subseteq Z(G)$.
Let $\chi \in \mathrm{Irr}\,(E)$ be faithful. Then χ is extendible to an irreducible character of G.

In Theorem (2.11) a similar result is proved as in Theorem (2.4), but now under the more general conditions (a) E is a normal $*$-subgroup of G, (e) $[E, K]Z(E)=E$, and (b), (c), (d), (f) as in the text of Theorem (2.4).

In Theorem (2.12) it is proved that any non-linear irreducible character of E can be extended to a character of G, where the finite groups G, K, H, E satisfy the conditions (b), (c), (d), (f), all as in Theorem (2.11), but now we are given only on E itself, that the order of the commutator subgroup of E is a prime number p and that $E/Z(E)$ is an elementary abelian p-group.

For correspondence theory of characters where $*$-groups are involved, see Theorem (2.16).

The interested reader is referred to Remark (2) after the proof of Theorem (2.12) for an exposition of the strategical line of proving like theorems from the past to the present.

Notations, conventions are mainly from Huppert's book [11] and from Isaacs' book [13]. Groups are always finite. The commutator subgroup of G is denoted by G' or by $[G, G]$.

QUOTED RESULTS, CHARACTER TRIPLE PROPERTIES

We recall some known theorems, properties and definitions for the convenience of the reader.

COROLLARY (6.17) of [13] (Gallagher). *Let $N \trianglelefteq G$ and let $\chi \in \mathrm{Irr}\,(G)$ such that $\chi|_N=\vartheta \in \mathrm{Irr}\,(N)$. Then the characters $\beta\chi$ for $\beta \in \mathrm{Irr}\,(G/N)$ are irreducible, distinct for distinct β and are all of the irreducible constituents of ϑ^G. In fact we have $\vartheta^G=\sum_{\beta \in \mathrm{Irr}\,(G/N)} \beta(1)(\beta\chi)$.* ♦

EXERCISE (6.3(a)) of [13]. Let $N \trianglelefteq G$ and let $\chi \in \mathrm{Irr}\,(G)$. Then we say that χ is *fully ramified with respect to G/N* if $\chi|_N=e\vartheta$ with $\vartheta \in \mathrm{Irr}\,(N)$ and $e^2=|G/N|$. Analogously, let $\eta \in \mathrm{Irr}\,(N)$. Then η is called *fully ramified with respect to G/N* if $\eta^G=f\alpha$ with $\alpha \in \mathrm{Irr}\,(G)$ and $f^2=|G/N|$. ♦

EXERCISE (6.3) of [13]. Let $N \trianglelefteq G$, $\vartheta \in \mathrm{Irr}\,(N)$, $\chi \in \mathrm{Irr}\,(G)$ with $[\chi|_N, \vartheta] \neq 0$. Then the following are equivalent.

(a) χ is fully ramified with respect to G/N,
(b) ϑ is fully ramified with respect to G/N,
(c) ϑ is invariant in G and χ vanishes on $G-N$,
(d) χ is the unique irreducible contituent of ϑ^G and ϑ is invariant in G. ♦

DEFINITION (page 186 of [13]). Let $N \trianglelefteq G$, $\vartheta \in \mathrm{Irr}(N)$ with ϑ invariant in G. Then (G, N, ϑ) is called a *character triple*.

Let us consider a character triple (G, N, ϑ) in case G/N is abelian, and suppose ϑ is linear. Then define $\langle\!\langle x, y\rangle\!\rangle_\vartheta = \vartheta(x^{-1}y^{-1}xy)$ for $x, y \in G$. Since ϑ is invariant in G, $\langle\!\langle .,. \rangle\!\rangle_\vartheta$ is constant on cosets of N in G. Thus we may view $\langle\!\langle\, , \rangle\!\rangle_\vartheta$ as being defined on G/N when convenient. This form is bilinear (in the multiplicative sense) and is alternating, that is $\langle\!\langle x, x\rangle\!\rangle_\vartheta = 1$ for all $x \in G$.

In case G/N is not abelian we may view $\langle\!\langle\, , \rangle\!\rangle_\vartheta : V \to \mathbb{C}$ as being defined on $V = \{(x, y) \in G \times G \mid [x, y] \in N\}$. We would like to be able to define it even when $\vartheta(1) > 1$. Isaacs [12] did the following. Suppose $x, y \in G$ and $[x, y] \in N$, and let ϑ be invariant in G. Then $\vartheta \in \mathrm{Irr}(N)$ can be extended to a character $\psi \in \mathrm{Irr}(\langle N, y\rangle)$ as $\langle N, y\rangle / N$ is cyclic ([11], Satz V.17.12). As x is an element of the normalizer of $\langle N, y\rangle$ in G, ψ^x is another extension of ϑ, so Gallagher's Theorem yields $\psi^x = \lambda\psi$ for a unique linear character λ of $\langle N, y\rangle$ with $N \subseteq \mathrm{Ker}(\lambda)$. Now set $\langle\!\langle x, y\rangle\!\rangle_\vartheta = \lambda(y)$, so $\psi(xyx^{-1}) = \langle\!\langle x, y\rangle\!\rangle_\vartheta \cdot \psi(y)$. It is not difficult to see that $\langle\!\langle x, y\rangle\!\rangle_\vartheta$ is well-defined, i.e. independent of the choice of $\psi \in \mathrm{Irr}(\langle N, y\rangle)$ extending ϑ. Although $\langle\!\langle\, , \rangle\!\rangle_\vartheta$ depends on ϑ the subscript will be dropped if it is clear from the context.

Under the abbreviation CTP (Character Triple Properties) we collect some (elementary) facts about the form $\langle\!\langle \cdot, \cdot \rangle\!\rangle_\vartheta$. For proofs, see [12].

CTP Let (G, N, ϑ) be a character triple. Then

(a) $\langle\!\langle x_1x_2, y\rangle\!\rangle = \langle\!\langle x_1, y\rangle\!\rangle \cdot \langle\!\langle x_2, y\rangle\!\rangle$, for $x_i, y \in G$, $[x_i, y] \in N$ $(i = 1, 2)$.
(b) $\langle\!\langle x, y_1y_2\rangle\!\rangle = \langle\!\langle x, y_1\rangle\!\rangle \cdot \langle\!\langle x, y_2\rangle\!\rangle$, for $x, y_j \in G$, $[x, y_j] \in N$ $(j = 1, 2)$.
(c) $\langle\!\langle x, y\rangle\!\rangle = \langle\!\langle y, x\rangle\!\rangle^{-1}$ for $x, y \in G$, $[x, y] \in N$.
(d) $\langle\!\langle x, x\rangle\!\rangle = 1$ for $x \in G$.
(e) $\langle\!\langle x, n\rangle\!\rangle = 1 = \langle\!\langle n, x\rangle\!\rangle$ for $n \in N$, $x \in G$.
(f) $\langle\!\langle nx, y\rangle\!\rangle = \langle\!\langle x, y\rangle\!\rangle = \langle\!\langle x, ny\rangle\!\rangle$ for $x, y \in G$, $[x, y] \in N$, $n \in N$.
(g) $\langle\!\langle x, y\rangle\!\rangle = \langle\!\langle \sigma(x), \sigma(y)\rangle\!\rangle$ for $x, y \in G$, $\sigma \in \mathrm{Aut}(G)$, $[x, y] \in N$, $N^\sigma = N$, $\vartheta^\sigma = \vartheta$.
(h) Let $H/N \subseteq Z(G/N)$. Put $H^\perp = \{x \in G \mid \langle\!\langle x, h\rangle\!\rangle = 1 \text{ for all } h \in H\}$. Then $H^\perp$ is a subgroup of G and $N \subseteq H^\perp$. We have $H^\perp \trianglelefteq G$.
(i) Let G/N be abelian. Then
  (α) ϑ is extendible to G if and only if $G^\perp = G$.
  (β) ϑ is fully ramified with respect to G/N if and only if $G^\perp = N$.
  (γ) Every irreducible constituent of ϑ^G is fully ramified with respect to $G/G^\perp$.
(j) Suppose $H/N \subseteq Z(G/N)$. Then $|G/H^\perp| \leq |H/N|$. If G/N is abelian and $G^\perp = N$, then equality holds and $H^{\perp\perp} = H$.
(k) Suppose G/N is abelian. Then
  (α) There is a unique subgroup M of G, maximal with respect to ϑ having a G-invariant extension to M.

(β) Every extension of ϑ to M is fully ramified with respect to G/M.
(γ) If $G \subseteq X$ with $G \trianglelefteq X$, $N \trianglelefteq X$ and ϑ invariant in X, then $M \trianglelefteq X$.

§ 1. THE STRUCTURE AND REPRESENTATION THEORY OF $*$-GROUPS

(1.1) DEFINITION. A group G is called a *$*$-group* if it satisfies the following.
(a) $Z(G)$ is cyclic,
(b) $Z(G) \supseteq [G, G]$,
(c) Every coset of $Z(G)$ in G contains an element x such that $Z(G) \cap \langle x \rangle = \{1\}$.

The concept of a $*$-group has been introduced by Isaacs (see Definition (4.1) in [12]). Examples of $*$-groups are the extra special p-groups of exponent p for odd primes p and the central product of an extra special 2-group with a cyclic group of order 4. We shall give another example of a $*$-group in Example (1.21).

Note that if G is a $*$-group, then the commutator map $[.,.]$ defines a multiplicative alternating form on $G/Z(G)$, which is non-degenerate. For fix $z \in Z(G)$ with $Z(G) = \langle z \rangle$ and $x, y \in G$. Define $f(\bar{x}, \bar{y}) = \zeta^i$ if $[x, y] = z^i$ (here $\bar{x}$ and $\bar{y}$ denote the canonical images of x and y in $G/Z(G)$ and $\zeta \in \mathbb{C}$ is chosen such that $\langle \zeta \rangle \cong Z(G)$). Then $f: G/Z(G) \times G/Z(G) \to \mathbb{C}$ defines a non-degenerate multiplicative alternating form. We set $Sp(G)$ for the group of all automorphisms of $G/Z(G)$ which preserve this form. If $G/Z(G)$ is elementary abelian of order p^n, then $Sp(G)$ is the ordinary symplectic group $Sp(n, p)$. We shall see below that in a $*$-group $|G/Z(G)|$ is always a square so that in the latter case n must be even.

(1.2) LEMMA. *Let G be a $*$-group and let $\sigma \in Sp(G)$. Then we have the following.*
(a) *There exists a $\tau \in \mathrm{Aut}\ (G)$ which is trivial on $Z(G)$ and induces σ on $G/Z(G)$.*
(b) *If $\tau \in \mathrm{Aut}\ (G)$ is trivial on $Z(G)$ and on $G/Z(G)$, then $\tau \in \mathrm{Inn}\ (G)$.*

PROOF. See ([12], Lemma (4.2)). ♦

(1.3) COROLLARY. *Let G be a $*$-group. Let $J = C_{\mathrm{Aut}(G)}(Z(G)) = \{\tau \in \mathrm{Aut}\ (G) | \tau(z) = z$ for all $z \in Z(G)\}$ and $I = \mathrm{Inn}\ (G) \trianglelefteq J$. Then*
(a) *$J/I \simeq Sp(G)$.*
(b) *If $|G/Z(G)|$ is odd, there exists a $\tau \in J$ such that τ inverts $G/Z(G)$, $\tau^2 = 1$, $J = IS$ and $I \cap S = \{1\}$, where $S = C_J(\tau)$.*

PROOF. There is a natural homomorphism from J into $Sp(G)$. Now Lemma (1.2)(a) asserts that this map is surjective and Lemma (1.2)(b) asserts that its kernel is I. We now have (a). Assume $|G/Z(G)|$ is odd. Then $Sp(G)$ contains a central involution σ, which inverts all elements of $G/Z(G)$, namely define $\sigma \in Sp(G)$ by $\sigma(\bar{x}) = \overline{x^{-1}} = \bar{x}^{-1}$ for all $\bar{x} \in G/Z(G)$. (Observe that $\sigma \neq 1$ since $|G/Z(G)|$ is odd.) Because $|G/Z(G)|$ is odd there is a (not necessarily central) involution $\tau \in J$, which maps to σ under the natural map. For let ϱ be such that ϱ maps to σ under the natural map then, because $\sigma^2 = 1$, we have $\varrho^2 \in I$, say $\varrho^2 = \phi_x$ for some $x \in G$. Now by the oddness of $|G/Z(G)|$ every element of $G/Z(G)$ is a square, say $\bar{x} = \bar{y}^2$, so $x = y^2 z$ for some $z \in Z(G)$. Then we have $\varrho^2 = \phi_{y^2 z} = \phi_{y^2}$. Now define $\tau = \phi_{y^{-1}} \cdot \varrho \in J$, which shows the above statement to be true. Now let $\psi \in J$ and τ the involution defined above. Then because σ is central $\psi\tau\psi^{-1}\tau^{-1}$ maps to 1 under the natural map. Hence $\psi\tau\psi^{-1} \in \tau I$, whence $I\langle\tau\rangle \trianglelefteq J$. Now let $P \in \mathrm{Syl}_2(I\langle\tau\rangle)$ with $\tau \in P$. By the Frattini argument we have $J = I\langle\tau\rangle N_J(P)$. Let $\upsilon \in N_J(P)$. Then $\upsilon\tau\upsilon^{-1} = \tau\pi$ for some $\pi \in P$. Now $\tau \neq 1$ so $\pi \neq \tau$ and we must have $\pi \in I$. But $|I|$ is odd so we conclude that $\pi = 1$. It follows that $J \supseteq I\langle\tau\rangle C_J(\tau) \supseteq I\langle\tau\rangle N_J(\tau) = J$. Thus $J = IS$ with $S = C_J(\tau)$. Now let $\phi_x \in I \cap S$, so $\tau\phi_x\tau^{-1} = \phi_x$ or equivalently $\tau(x)g\tau(x)^{-1} = xgx^{-1}$ for all $g \in G$. So $x^{-1}\tau(x) \in Z(G)$ and hence $\bar{x} = \overline{\tau(x)} = \sigma(\bar{x}) = \bar{x}^{-1}$, so $\bar{x}^2 = \bar{1}$, but $|G/Z(G)|$ is odd thus $\bar{x} = \bar{1}$ and hence $x \in Z(G)$. So afterall $\phi_x = 1$. This implies $I \cap S = \{1\}$. This proves (1.3). ♦

(1.4) THEOREM. *Let $K \trianglelefteq G$ and assume that K is a $*$-group and*

$$L = Z(K) \subseteq Z(G)$$

with $|G/Z(G)|$ odd. Then there exists a unique conjugacy class of subgroups $U \subseteq G$ satisfying

(a) *$G = UK$ and $U \cap K = L$.*

(b) *$U \supseteq C_G(K)$.*

(c) *There exists a $\tau \in \mathrm{Aut}\,(G)$ such that $\tau^2 = 1$, τ inverts K/L and $U = C_G(\tau)$.*

PROOF. We give here a slightly elaborate proof for the convenience of the reader in respect to the original one in ([12], Corollary (4.4)).

Let $C = C_G(K)$ so that $C \trianglelefteq G$ (because $K \trianglelefteq G$) and G/C is naturally isomorphic to a subgroup $X \subseteq J = C_{\mathrm{Aut}\,(K)}(L)$ (here we use $L \subseteq Z(G)$). Since KC/C maps onto $I = \mathrm{Inn}\,(K)$ under this natural map, we have $X \subseteq I$ and hence by Corollary (1.3) $X = I(S \cap X)$, where $S = C_J(\varrho)$, $\varrho^2 = 1$ and ϱ inverts K/L.

Let U/C be the inverse image of $S \cap X$ in G/C so that $U(KC) = G$ and $U \cap KC = C$ (since $I \cap S = \{1\}$). We have then $U \supseteq C$ and $G = UK$.

Also $L = K \cap C \subseteq K \cap U = U \cap (KC \cap K) = K \cap C = L$ and hence U satisfies (a) and (b).

We define τ on $G = UK$ by $\tau(uk) = u\varrho(k)$ for $u \in U$ and $k \in K$. Because ϱ is trivial on $L = U \cap K$, τ is well-defined. By definition of U conjugation by elements of U commutes with ϱ on K, so

$$\tau(u_1k_1)\tau(u_2k_2) =$$

$$u_1\varrho(k_1)u_2\varrho(k_1) =$$

$$u_1u_2u_2^{-1}\varrho(k_1)u_2\varrho(k_2) =$$

$$u_1u_2(\phi_{u_2^{-1}} \cdot \varrho)(k_1)\varrho(k_2) =$$

$$u_1u_2(\varrho \cdot \phi_{u_2^{-1}})(k_1)\varrho(k_2) =$$

$$u_1u_2\varrho(u_2^{-1}k_1u_2)\varrho(k_2) =$$
$$u_1u_2\varrho(u_2^{-1}k_1u_2k_2) =$$
$$\tau(u_1u_2u_2^{-1}k_1u_2k_2) = \tau(u_1k_1u_2k_2) \text{ for all } u_1, u_2 \in U \text{ and } k_1, k_2 \in K.$$

It follows easily that $\tau \in \mathrm{Aut}\ (G)$ and to conclude

$$C_G(\tau) = \{g \in G \mid \tau(g) = g\} = \{uk \mid u\varrho(k) = uk\} = UC_K(\varrho).$$

But $C_K(\varrho)C/C$ is mapped within $S \cap X$, so $UC_K(\varrho) = U$. This proves (c).

Now let $V \subseteq G$ satisfy (a), (b) and (c). We must show that V is conjugate to U in G. Since $V \supseteq C$, V is the inverse image in G of a subgroup $Y \subseteq X$. By (c) it follows that Y centralizes some $\tau_0 \in J$, which inverts K/L and with $t_0^2 = 1$. It follows that $\tau_0 \in \tau I$ and thus τ_0 is conjugate to τ by Sylow's theorem in $I\langle\tau\rangle$. Since $Y = C_X(\tau_0)$ it follows that Y is conjugate to $S \cap X = C_X(\tau)$ via an element of $I \subseteq X$ and hence $V^k = U$ for some $k \in K$. This completes the proof of (1.4). ♦

In order to develop some of the representation theory of $*$-groups we need some structure theorems on $*$-groups. Observe that Definition (1.1)(b) yields the nilpotency of any $*$-group. By Definition (1.1)(a) an abelian $*$-group must be cyclic. It turns out that we do not need Definition (1.1)(c) for the representation theory of a $*$-group.

First we state two properties which are well-known.

(1.5) PROPOSITION. *Let G be a nilpotent group. Then G has a faithful irreducible character if and only if $Z(G)$ is cyclic.*

PROOF. This is a corollary to Theorem (2.32) of [13]. ♦

(1.6) LEMMA. *Let $\chi \in \mathrm{Irr}\ (G)$ be faithful. Assume that $[G, G] \subseteq Z(G)$. Then $\chi(1)^2 = |G/Z(G)|$ and χ vanishes off $Z(G)$.*

PROOF. See ([13], page 28). ♦

REMARK. Assume the hypothesis of Lemma (1.6). Then $Z(G)$ must be cyclic by Proposition (1.5). Further if χ is linear, then because χ is faithful G/G' is cyclic by Proposition (1.5). So then $G/Z(G)$ is cyclic whence G abelian.

Next we prove two theorems in which the behaviour of $*$-groups in direct products and in normal subgroups of $*$-groups is revealed.

(1.7) THEOREM. *Let G_1 and G_2 be $*$-groups. Then $G = G_1 \times G_2$ is a $*$-group if and only if* $\gcd\ (|Z(G_1)|, |Z(G_2)|) = 1$.

PROOF. The "only if" part is trivial by a well known theorem on cyclic groups. So let us prove the "if" part. The only thing to check is Definition (1.1)(c) for the group G. So let $(x_1, x_2)Z(G)$ be a coset of $Z(G)$ in G. Now $Z(G) = Z(G_1) \times Z(G_2)$ hence $(x_1, x_2)Z(G) = x_1Z(G_1) \times x_2Z(G_2)$. Now there exist

$y_i \in G_i$ with $y_i \in x_i Z(G_i)$ and $Z(G_i) \cap \langle y_i \rangle = \{1\}$ $(i=1,2)$. But then $(y_1, y_2)Z(G) = (x_1, x_2)Z(G)$ and $Z(G) \cap \langle (y_1, y_2) \rangle = \{1\}$. This proves (1.7). ♦

(1.8) THEOREM. *Let G be a $*$-group. Let p be a prime such that $p \mid |G|$ and let $P \in \mathrm{Syl}_p(G)$. Then $P \trianglelefteq G$ and P is $*$-group.*

PROOF. By Definition (1.1)(b) G is nilpotent so every Sylow subgroup of G is a normal subgroup of G. Now write $G = P_1 \times \cdots \times P_t$, where $P_i \in \mathrm{Syl}_{p_i}(G)$ and $\{p_1, \ldots, p_t\}$ are the different prime divisors of $|G|$. Now we have $Z(G) = Z(P_1) \times \cdots \times Z(P_t)$ so for all $i = 1, \ldots, t$ $Z(P_i)$ is cyclic. Further $G/Z(G) = P_1/Z(P_1) \times \cdots \times P_t/Z(P_t)$ so for all $i = 1, \ldots, t$ $P_i/Z(P_i)$ is abelian. We prove that P_1 is a $*$-group. Let $x \in P_1$ and consider the coset $xZ(P_1)$ of $Z(P_1)$ in P_1. Now $(x, 1, \ldots, 1)Z(G) = xZ(P_1) \times Z(P_2) \times \cdots \times Z(P_t)$ is a coset of $Z(G)$ in G. So there exists a $(y_1, \ldots, y_t) \in G$ with $(y_1, \ldots, y_t) \in (x, 1, \ldots, 1)Z(G)$ and $\langle (y_1, \ldots, y_t) \rangle \cap Z(G) = \{1\}$. Then $(y_1 x^{-1}, y_2, \ldots, y_t) \in Z(G)$ so $y_1 \in xZ(P_1)$ and $y_i \in Z(P_i)$ $(i = 2, \ldots, t)$. Since $Z(G) \cap \langle (y_1, \ldots, y_t) \rangle = \{1\}$ we see that $y_i = 1$ for all $i = 2, \ldots, t$ and $Z(P_1) \cap \langle y_1 \rangle = \{1\}$. Hence P_1 is a $*$-group and analogously one proves that P_i is a $*$-group for $i = 2, \ldots, t$. This proves (1.8). ♦

We shall need the following two results in an essential way.

(1.9) LEMMA. (Seitz) *Let $\chi \in \mathrm{Irr}\,(G)$. Let $H \leq G$ such that $\chi = \lambda^G$ for some $\lambda \in \mathrm{Irr}\,(H)$ with $\lambda(1) = 1$. Let $N \trianglelefteq G$ be abelian. Then there exists a $K \leq G$ with*
(a) *$N \subseteq K \subseteq G$.*
(b) *$\chi = \mu^G$ for some $\mu \in \mathrm{Irr}\,(K)$ with $\mu(1) = 1$.*

PROOF. See ([13], Exercise (6.11)). ♦

(1.10) THEOREM. (Shoda) *Let G be a non-abelian group. Assume that $[G, G] \subseteq Z(G)$ and $Z(G)$ is cyclic. Let M be a maximal abelian normal subgroup of G and let $\chi \in \mathrm{Irr}\,(G)$ be faithful. Then $\chi(1) = |G/M|$.*

PROOF. Note that $MZ(G)$ is abelian and normal, so we have $G' \subseteq Z(G) \subseteq M$. By Proposition (1.5) G has indeed faithful irreducible characters. As G is nilpotent, G is an M-group, whence χ is monomial. According to Lemma (1.9) there exists $M \subseteq K \subseteq G$ and a $\lambda \in \mathrm{Irr}\,(K)$ with $\lambda(1) = 1$ and $\chi = \lambda^G$. Now $\{1\} = \ker\,(\chi) = \ker\,(\lambda^G) = \mathrm{core}_G(\ker\,(\lambda))$. As $K' \subseteq \ker\,(\lambda)$ and $G' \subseteq K$, it follows first that $K \trianglelefteq G$ and then $K' \trianglelefteq G$, whence that $K' \leq \mathrm{core}_G(\ker\,(\lambda)) = \{1\}$. Hence K is abelian. By the choice of M we now have $M = K$, so $\chi(1) = \lambda^G(1) = |G/M|$. This proves (1.10). ♦

In Theorem (1.11) $\phi(n)$ stands for the Euler ϕ-function of the positive integer n. Although the contents of Theorem (1.11) are known, the proof of it is perhaps not often seen. So we give it here.

(1.11) THEOREM. *Let G be a non-abelian group. Assume that $[G, G] \subseteq Z(G)$ and $Z(G)$ is cyclic. Then G has $\phi(|Z(G)|)$ irreducible faithful characters, which* *are described as follows. Let M be a maximal abelian normal subgroup of G and let λ be a faithful linear character of $Z(G)$. Let μ be an extension of λ to M. Then μ^G is a faithful irreducible character of G. There is a one-to-one correspondence between the faithful linear characters of $Z(G)$ and the faithful irreducible characters of G. Moreover, if $\chi \in \mathrm{Irr}\,(G)$ is faithful, then $\chi(1) = \sqrt{|G/Z(G)|}$.*

PROOF. Let $\lambda \in \mathrm{Irr}\,(Z(G))$ be faithful. By Frobenius reciprocity and Clifford's Theorem we can choose a $\mu \in \mathrm{Irr}(M)$ such that $\mu|_{Z(G)} = \lambda$. We claim that $\mu^G \in \mathrm{Irr}\,(G)$. For let χ be an irreducible constituent of λ^G, such that $[\chi, \mu^G] \neq 0$. Assume that $\ker(\chi) \supsetneqq \{1\}$. As G is nilpotent we have $\ker(\chi) \cap Z(G) = \ker(\lambda) \supsetneqq \{1\}$, a contradiction. So χ is faithful. And by Shoda's Theorem (1.10) we have $\chi(1) = |G/M| = \mu^G(1)$. Hence $\chi = \mu^G$, which proves the claim. Moreover, it follows that $\mu^G = \chi$ vanishes off M, as $M \trianglelefteq G$. Now $Z(G) \subseteq M \subseteq G$. Hence

$$
\begin{aligned}
1 = [\chi, \chi] = [\mu^G, \mu^G] &= \frac{1}{|G|} \sum_{g \in G} |\chi(g)|^2 \geq \\
&\geq \frac{1}{|G|} \Big(\sum_{g \in M - Z(G)} |\chi(g)|^2 + \sum_{g \in Z(G)} |\chi(g)|^2 \Big) = \\
&= \frac{1}{|G|} \Big(\sum_{g \in M - Z(G)} |\chi(g)|^2 + \chi(1)^2 \cdot \sum_{g \in Z(G)} |\lambda(g)|^2 \Big) = \\
&= \frac{1}{|G|} \sum_{g \in M - Z(G)} |\chi(g)|^2 + \frac{\chi(1)^2 |Z(G)|}{|G|} = \\
&= \frac{1}{|G|} \sum_{g \in M - Z(G)} |\chi(g)|^2 + 1,
\end{aligned}
$$

where the last equality follows by Lemma (1.6). It follows that χ vanishes off $Z(G)$. In fact we have

$$
\mu^G(g) = \begin{cases} 0 & \text{if } g \notin Z(G) \\ \lambda(g)|G/M| & \text{if } g \in Z(G). \end{cases}
$$

So μ^G does not depend on μ or M (note that by Shoda's Theorem all maximal abelian normal subgroups of G must have the same order), but uniquely on λ only.

Because $Z(G)$ is cyclic the number of faithful linear characters of $Z(G)$ equals the number of generators of $Z(G)$, which is $\phi(|Z(G)|)$.

Finally, we claim that if $\chi_1, \chi_2 \in \mathrm{Irr}\,(G)$ are faithful, then $\chi_1 = \chi_2$ if and only if $[\chi_1|_{Z(G)}, \chi_2|_{Z(G)}] \neq 0$. For the "only if" part is trivial. So assume that $[\chi_1|_{Z(G)}, \chi_2|_{Z(G)}] \neq 0$, i.e. $\chi_1|_{Z(G)}$ and $\chi_2|_{Z(G)}$ have a common irreducible constituent $\lambda \in \mathrm{Irr}\,(Z(G))$. Now by Lemma (1.6) χ_1 and χ_2 vanish off $Z(G)$. So we only have to show that $\chi_1|_{Z(G)} = \chi_2|_{Z(G)}$. Note that by Lemma (1.6) $\chi_1(1) = \chi_2(1) = \sqrt{|G/Z(G)|}$. So $\chi_1|_{Z(G)} = \chi_1(1)\lambda = \chi_2(1)\lambda = \chi_2|_{Z(G)}$. Hence $\chi_1 = \chi_2$. This completes the proof of (1.11). ♦

(1.12) PROPOSITION. *Let G be a $*$-group of odd order. Then the following are equivalent.*

(a) *G is an extra special p-group of exponent p for some prime p.*

(b) *All the non-linear irreducible characters of G are faithful and $[G,G] = Z(G) = \Phi(G)$.*

PROOF. (a)$\Rightarrow$(b). This follows from the representation theory of extra special p-groups (of exponent p), as a corollary to Theorem (1.11).

(b)$\Rightarrow$(a) If all the non-linear irreducible characters of G are faithful, then by Theorem (1.11) and a theorem of Burnside it follows that

$$|G| = |G/G'| + \phi(|Z(G)|)\cdot|G/Z(G)|.$$

Now $|Z(G)| = |G'|$ so it follows that $\phi(|Z(G)|) = |Z(G)| - 1$. So $|Z(G)|$ must be a prime number p. By the nilpotency of G it follows that G is a p-group. Now G is a $*$-group of odd order and hence the classification of extra special p-groups (see ([11], page 355)) gives the desired result. This proves (1.12). ♦

(1.13) PROPOSITION. *Let G be a $*$-group. Let $x \in G - Z(G)$. Then there exists a $z \in Z(G)$ such that $G/C_G(x) \simeq \langle xz \rangle$.*

PROOF. Let $g \in G$ arbitrary. Then by Definition (1.1)(b) the commutator maps

$$[\cdot, g] : G \to G' \text{ and } [g, \cdot] : G \to G'$$

are homomorphisms both with kernel $C_G(g)$ and image $[g, G]$. So $G/C_G(g) \cong [g, G]$. Now let $x \in G - Z(G)$. By Definition (1.1)(c) there exists a $y \in G$ such that $y \in xZ(G)$ and $Z(G) \cap \langle y \rangle = \{1\}$. Now $C_G(x) = C_G(y)$ and by Definition (1.1)(a) $[y, G]$ is cyclic, say $[y, G] = \langle [y, g] \rangle$ for some $g \in G$. We show that order (y) = order $([y, g])$.

Write order $(y) = k$ and order $([y, g]) = l$.

Now $[y, g]^k = [y^k, g] = [1, g] = 1$. So $l | k$. Let $h \in G$ arbitrary. Then $[y, h] = [y, g]^m$ for some $m \in \mathbb{Z}$. So $[y^l, h] = [y, h]^l = [y, g]^{ml} = 1^m = 1$. So $y^l \in Z(G)$ because h was arbitrarily chosen. But $Z(G) \cap \langle y \rangle = \{1\}$, so $y^l = 1$. So $k | l$. This proves (1.13). ♦

The following result will be of importance in section 2.

(1.14) THEOREM. *Let G be a $*$-group. Let $H \leq G$ with $Z(H) = Z(G)$. Then*

(a) *G is the central product of H and $C_G(H)$ with $Z(H) = Z(C_G(H))$ amalgamated.*

(b) *H and $C_G(H)$ are both $*$-groups.*

PROOF. We have $H' \subseteq G' \subseteq Z(G) = Z(H)$. Also $Z(H)$ is cyclic because $Z(G)$ is. Now let $h \in H$. Then the coset $hZ(H)$ equals $hZ(G)$. So there exists an element $x \in G$ with $Z(G) \cap \langle x \rangle = \{1\}$ and $xZ(G) = hZ(G)$. It follows that $x \in H$ and H is a $*$-group. Now observe that $H \trianglelefteq G$ because $G' \subseteq H$. Let $g \in G$ arbitrary. Then the inner automorphism ϕ_g induces an automorphism τ of H. Clearly τ is trivial on $Z(H)$ as $Z(H)=Z(G)$. Also for all $h\in H$ we have $h^g=[g,h^{-1}]h$ and $[g,h^{-1}]\in G'\subseteq Z(H)$. So τ is also trivial on $H/Z(H)$. By Lemma (1.2)(b) we conclude that $\tau\in \mathrm{Inn}(H)$, which means, there exists an $h\in H$ such that $gxg^{-1}=hxh^{-1}$ for all $x\in H$. It follows that $h^{-1}g\in C_G(H)$ and so $g\in HC_G(H)$. We now have $G=HC_G(H)$. Furthermore, we have $[H,C_G(H)]=\{1\}$ and $C_G(H)\cap H=C_H(H)=Z(H)=Z(G)$.

Now let $z\in Z(C_G(H))$, so $z\in C_G(H)$ and $[z,c]=1$ for all $c\in C_G(H)$. Let $g\in G$, say $g=hc$ for some $h\in H$ and $c\in C_G(H)$. Then $zg=zhc=hzc=hcz=gz$. Hence $z\in Z(G)$. On the other hand $Z(G)=Z(H)\subseteq C_G(H)$ so $Z(G)\subseteq Z(C_G(H))$. We now have $Z(G)=Z(H)=Z(C_G(H))$. It is now clear that $C_G(H)$ is also a $*$-group by the same argument above for H. This proves (1.14). ♦

We shall need the following results on character triples for later purposes. See the Introduction for the definitions and some specific properties of characters triples.

(1.15) DEFINITION. Let (G,N,θ) and (H,M,ϕ) be character triples. Let $T: G/N\to H/M$ be an isomorphism and if $N\subseteq U\subseteq G$ and $T(U/N)=V/M$, write $T(U)=V$. If $\psi\in \mathrm{Char}\ (U/N)$, denote the corresponding character of V/M by $T(\psi)$.

Let $F: \mathrm{Char}\ (U|\theta)\to \mathrm{Char}\ (V|\phi)$ be defined for every U with $T(U)=V$. Then (T,F) is called an *isomorphism of character triples* provided that for each U the following hold.

(a) $F(\chi_1+\chi_2)=F(\chi_1)+F(\chi_2)$ for $\chi_1,\chi_2\in \mathrm{Char}\ (U|\theta)$.
(b) $F(\psi\chi)=T(\psi)F(\chi)$ for $\chi\in \mathrm{Char}\ (U|\theta)$ and $\psi\in \mathrm{Char}\ (U/N)$.
(c) $F(\chi|_W)=F(\chi)|_{T(W)}$ for $\chi\in \mathrm{Char}\ (U|\theta)$ and $N\subseteq W\subseteq U$.
(d) $[F(\chi_1),F(\chi_2)]=[\chi_1,\chi_2]$ for $\chi_1,\chi_2\in \mathrm{Char}\ (U|\theta)$.
(e) $F(\chi^g)=F(\chi)^h$ for $\chi\in \mathrm{Char}\ (U|\theta)$, where $g\in G$ and $h\in T(gN)$.

In (e) above, $\chi^g\in \mathrm{Char}\ (U^g|\theta)$ is defined by $\chi^g(u^g)=\chi(u)$ as usual. In order to define a function F as in Definition (1.15) it suffices, by (a), to specify $F(\chi)$ for $\chi\in \mathrm{Irr}\ (U|\theta)$. By (d) F maps $\mathrm{Irr}\ (U|\theta)$ injectively to $\mathrm{Irr}\ (V|\phi)$, where $V=T(U)$. We claim that this map is also surjective. Clearly $F(\theta)=\phi$ because by (d) $[\theta,\theta]=[F(\theta),F(\theta)]$ and $F(\theta)\in \mathrm{Char}\ (M|\phi)$, that is $[\phi,F(\theta)]\neq 0$. Thus if $\chi\in \mathrm{Irr}\ (U|\theta)$ with $\chi|_N=e_\chi\theta$ we have $F(\chi)|_M=e_\chi\phi$ by (c) and (a). It follows that $\chi(1)/\theta(1)=F(\chi)(1)/\phi(1)$, so $\chi(1)\cdot\phi(1)=F(\chi)(1)\cdot\theta(1)(*)$.

Now if $\theta^U=\sum_{\chi\in \mathrm{Irr}\ (U|\theta)} e_\chi\chi$, then $|U:N|=\sum_{\chi\in \mathrm{Irr}\ (U|\theta)} e_\chi^2$. So we have

$$\phi(1)^2\cdot \sum_{\chi\in \mathrm{Irr}\ (U|\theta)} \chi(1)^2=\phi(1)^2\cdot\theta(1)^2\cdot|U:N|=$$

$$\theta(1)^2\cdot\phi(1)^2\cdot|V:M|=\theta(1)^2\cdot \sum_{\xi\in \mathrm{Irr}\ (V|\phi)} \xi(1)^2 \text{ (where } T(U)=V).$$

By $(*)$ it follows that $\sum_{\chi\in \mathrm{Irr}\ (U|\theta)} (F(\chi)(1))^2=\sum_{\xi\in \mathrm{Irr}\ (V|\phi)} \xi(1)^2$ and it follows that all $\xi\in \mathrm{Irr}\ (V|\phi)$ are of the form $F(\chi)$, so F maps onto $\mathrm{Irr}\ (V|\phi)$.

It is now clear how to define the isomorphism (T^{-1}, F^{-1}) from (H, M, ϕ) to (G, N, θ) and it follows that isomorphism is an equivalence relation. If (G, N, θ) and (H, M, ϕ) are isomorphic character triples, it follows from (e) and (b) above that the forms $\langle\!\langle\ ,\ \rangle\!\rangle_\theta$ and $\langle\!\langle\ ,\ \rangle\!\rangle_\phi$, defined on commuting pairs of elements of G/N and H/M agree under the isomorphism T. A trivial, though useful, example of an isomorphism of character triples is $(G, N, \theta) \simeq (G/K, N/K, \theta)$ if $K \trianglelefteq G$ and $K \subseteq \ker(\theta)$, where, as usual, we have identified θ with the corresponding character of N/K. Here if $\chi \in \mathrm{Char}\,(U|\theta)$, for $N \subseteq U \subseteq G$, then $K \subseteq \ker(\chi)$ and we let $F(\chi)$ be the naturally corresponding character of U/K.

The following theorem is essentially classical. What is really needed is little more than Schur's theory of projective representations and covering groups. (A stronger, choicefree, version appears in [4].)

(1.16) THEOREM. *Let (G, N, θ) be a character triple. Then there exists an isomorphic character triple (G^*, N^*, θ^*) such that*

(a) *θ^* is linear and faithful.*

(b) *Each coset of N^* in G^* contains an element x such that $N^* \cap \langle x\rangle = \{1\}$.*

PROOF. This is Theorem (8.2) in [12]. ♦

(1.17) COROLLARY. *Let (G, N, θ) be a character triple. Then there exists an isomorphic character triple (G^*, N^*, θ^*) such that*

(a) *θ^* is faithful and $\theta^*(1) = 1$.*

(b) *N^* is a cyclic group with $|N^*| = |G|\theta(1)$.*

(c) *$N^* \subseteq Z(G^*)$ and $G/N \simeq G^*/N^*$.*

(d) *$\chi(1)/\theta(1) = \chi^*(1)$ for all $N \subseteq U \subseteq G$ and $\chi \in \mathrm{Char}\,(U|\theta)$.*

PROOF. Note that if $(T, F) : (G, N, \theta) \to (G^*, N^*, \theta^*)$ is an isomorphism of the character triples, we have denoted $F(\chi)$ by χ^* in (d). Now (d) follows from the remarks made after Definition (1.15). Note that in (a) $Z(\theta^*) = N^*$ because θ^* is linear. But θ^* is faithful so $Z(\theta^*)$ is cyclic. Furthermore, everything is clear by Theorem (1.16). This proves (1.17). ♦

We give the following two known results as an illustration.

(1.18) COROLLARY. *Let (G, N, θ) be a character triple and let A act on G. Assume that N is A-invariant and θ is A-fixed. Then an isomorphic character triple (G^*, N^*, θ^*) exists, satisfying*

(a) *and* (b) *of Theorem* (1.16) *and also satisfying*

(c) *A acts on G^*, N^* is A-invariant and θ^* is A-fixed and such that the given isomorphism $T : G/N \to G^*/N^*$ is an A-isomorphism.*

PROOF. Let $\Gamma = G \rtimes A$. Apply Theorem (1.16) to the character triple (Γ, N, θ) to obtain $(\Gamma^*, N^*, \theta^*)$ with isomorphism (T, F). Define $G^* := T(G) \trianglelefteq \Gamma^*$. Now $N^* \subseteq Z(\Gamma^*)$ and thus $T(AN)/N^* \simeq AN/N \simeq A$ and acts on G^* in the desired manner. (Note that this argument actually proves more. For instance, if $\chi \in \mathrm{Char}\,(G|\theta)$ and $a \in A$, then $F(\chi^a) = F(\chi)^a$.) This proves (1.18). ♦

(1.19) LEMMA. *Let* $N \trianglelefteq G$, $\chi \in \mathrm{Irr}\,(G)$ *and* $\theta \in \mathrm{Irr}\,(N)$ *with* $[\chi|_N, \theta] \neq 0$. *Then* $\chi(1)/\theta(1) \in \mathbb{N}$ *and* $\chi(1)/\theta(1) | \gcd\,(\chi(1), |G/N|)$.

PROOF. Let $T = I_G(\theta)$ and so there exists $\psi \in \mathrm{Irr}\,(T)$ such that $\psi^G = \chi$ and $[\psi|_N, \theta] \neq 0$. Now consider the character triple (T, N, θ). Let (T^*, N^*, θ^*) be an isomorphic character triple with $\theta^*(1) = 1$ and θ^* faithful, by Theorem (1.16). Now by Corollary (1.17) $\psi(1)/\theta(1) = \psi^*(1)$. But $N^* \subseteq Z(T^*) \subseteq Z(\psi^*)$ and (see Theorem (3.12) in [13]) $\psi^*(1) \,\big|\, |T^* : Z(\psi^*)| \,\big|\, |T^*/N^*|$ and $T/N \simeq T^*/N^*$. So $\chi(1)/\theta(1) = (\psi(1)/\theta(1))|G:T| = \psi^*(1)|G:T| \,\big|\, |G:T| \cdot |T/N|$. This proves (1.19). ♦

Now the connection between $*$-groups and character triples is the following. Let (G, N, θ) be a character triple with G/N abelian and θ fully ramified with respect to G/N (see CTP i) for some equivalent conditions). By Theorem (1.16) there exists an isomorphic triple (G^*, N^*, θ^*) such that θ^* is linear and faithful, $G/N \simeq G^*/N^*$, N^* is cyclic and $N^* \subseteq Z(G^*)$ and each coset of N^* contains an element x such that $N^* \cap \langle x \rangle = \{1\}$. By Definition (1.15)(a) θ^* is also fully ramified with respect to G^*/N^*, say $(\theta^*)^{G^*} = e\chi^*$, where $\theta^G = e\chi$ and $e^2 = |G/N| = |G^*/N^*|$. Note that because $\theta^*(1) = 1$ we have $\chi^*(1) = e$. We claim that G^* is a $*$-group. As also χ^* is fully ramified with respect to G^*/N^* (see Exercise (6.3) of [13]), χ vanishes off N^*. Now $Z(G^*) \subseteq Z(\chi^*)$. Take $z \in Z(G^*)$. Then $|\chi^*(z)| = \chi^*(1) = e \neq 0$. So $\chi^*(z) \neq 0$ and hence $z \in N^*$. It follows that $N^* = Z(G^*)$ and thus $Z(G^*)$ is cyclic. Also $G^*/N^* = G^*/Z(G^*)$ is abelian so $[G^*, G^*] \subseteq Z(G^*)$. The claim now follows from the above properties of the character triple (G^*, N^*, θ^*). Now that in fact we have $\chi^*(1) = \sqrt{|G^*/Z(G^*)|}$.

We shall encounter the above situation again in § 2.

(1.20) LEMMA. (Isaacs) *Let* $N \trianglelefteq G$. *Let* $\theta \in \mathrm{Irr}\,(N)$ *be invariant in* G. *Suppose that for every prime* p *dividing* $|G/N|$ θ *is extendible to* PN *with* $P \in \mathrm{Syl}_p(G)$. *Then* θ *is extendible to* G.

PROOF. Let (G^*, N^*, θ^*) be a character triple isomorphic to the character triple (G, N, θ) by Theorem (1.16). We use $*$ to denote appropriate images. If $N \subseteq U \subseteq G$ and $N^* \subseteq U^* \subseteq G^*$, then by Corollary (1.17)(d) and the remarks made after Definition (1.15) θ is extendible to U if and only if θ^* is extendible to U^* (note that extendability of θ to U is equivalent to the existence of a $\chi \in \mathrm{Irr}\,(U|\theta)$ with $\chi(1)/\theta(1) = 1$ and similarly for the extendability of θ^* to U^*). Thus θ^* is extendible to the inverse image in G^* of every Sylow p-subgroup of G^*/N^*, for which $p \,\big|\, |G^*/N^*|$. Now write $\theta^* = \lambda$. And let $m = \mathrm{order}\,(\lambda)$ (in the group of linear characters of N^* of course). For each $p|m$ we may choose λ_p, a power of λ, such that $\lambda = \prod_{p|m} \lambda_p$ and order (λ_p) is a power of p. We shall show that λ_p is extendible to an irreducible character μ_p of G^*. Then $\mu = \prod_{p|m} \mu_p$ is an extension of λ.

Since λ_p is a power of λ and λ is extendible to P^*N^* with $P^* \in \mathrm{Syl}_p(G^*)$, it follows that λ_p is extendible to P^*N^*. Also λ_p is invariant in G^*. We now see that it is no loss to assume that m is a power of p.

Let υ be an extension of λ to P^*N^*. Since $p \nmid |G^*:P^*N^*|$ and $\upsilon(1)=1$ as $\lambda(1)=\theta^*(1)=1$, it follows that $p \nmid \upsilon^{G^*}(1)$ and hence there exists an irreducible constituent χ of υ^{G^*} with $p \nmid \chi(1)$. Let $\chi(1)=f$. We have $[\chi|_{P^*N^*}, \upsilon] \neq 0$ and hence $[\chi|_{N^*}, \lambda] \neq 0$. Since λ is invariant in G^*, we conclude that $\chi|_{N^*}=f\lambda$ and thus $\det(\chi)|_{N^*}=\lambda^f$. Let $\delta=\det(\chi)$. Since $p \nmid f$ and m is a power of p, we may choose $b \in \mathbb{Z}$ with $bf \equiv 1 \pmod m$. Then we have $\delta^b|_{N^*}=\lambda^{bf}=\lambda$. This proves (1.20). ♦

We end § 1 with the example of a $*$-group as promised in the beginning of this section.

(1.21) EXAMPLE. Let p be a prime. Let G of order p^6 be given by a presentation

$$G=\langle a,b,c \mid a^{p^2}=b^{p^2}=c^{p^2}=1, c=[a,b], [a,c]=[b,c]=1 \rangle.$$

We will show in a number of steps that G is a $*$-group. Observe that $c \in Z(G)$. Further $a^{-1}b^{-1}ab=c$, so $ab=bac$. In what follows i, j and k denote integers.

(i) $\quad a^ib=ba^ic^i$.

PROOF. $b^{-1}ab=ac$, hence $b^{-1}a^ib=a^ic^i$. ♦

(ii) $\quad a^ib^j=b^ja^ic^{ij}$.

PROOF. Use induction on j. For $j=1$ this is (i). Now $a^ib^{j+1}=a^ib \cdot b^j = ba^ic^ib^j=b\cdot a^ib^j\cdot c^i=b\cdot b^ja^ic^{ij}\cdot c^i=b^{j+1}a^ic^{i(j+1)}$. ♦

Note that by (ii) it follows that $G=\{a^ib^jc^k \mid i,j,k \in \{0,\ldots,p^2-1\}\}$

(iii) $\quad (a^ib^j)^k=a^{ik}b^{jk}c^{-(1/2)ijk(k-1)}$.

PROOF. Use induction on k. For $k=1$ there is nothing to prove. Now

$$\begin{aligned}
&(a^ib^j)^{k+1}=\\
&(a^ib^j)^ka^ib^j=\\
&a^{ik}b^{jk}c^{-(1/2)ijk(k-1)}\cdot a^ib^j=\\
&a^{ik}b^{jk}\cdot b^ja^ic^{ij}\cdot c^{-(1/2)ijk(k-1)}=\\
&a^{ik}\cdot b^{j(k+1)}a^ic^{ij}c^{-(1/2)ijk(k-1)}=\\
&a^{ik}a^ib^{j(k+1)}c^{-ij(k+1)}c^{ij}c^{-(1/2)ijk(k-1)}=\\
&a^{i(k+1)}b^{j(k+1)}c^{-(1/2)ijk(k+1)}. \quad ♦
\end{aligned}$$

(iv) $\quad$ If $a^ib^j \in Z(G)$, then $p^2 \mid i$ and $p^2 \mid j$.

PROOF. Let $a^ib^j \in Z(G)$. Then $ba^ib^j = a^ib^{j+1}$. So by (i) we have $a^ib^{j+1}c^{-i} = a^ib^{j+1}$. Hence $c^i = 1$ and thus $p^2|i$. By a similar argument we have $p^2|j$. ♦

Now it follows that $Z(G) = \langle c \rangle = G' \simeq C_{p^2}$. One easily checks that $G/G' \simeq C_{p^2} \times C_{p^2}$. Also $\exp(G) = p^2$. A non-trivial coset of $Z(G)$ in G has the form $a^ib^jZ(G)$ with i and j not both zero and $i, j \in \{0, \ldots, p^2-1\}$. Now let $x = a^ib^jc^{ij}$. Then clearly we have $x \in a^ib^jZ(G)$. Suppose $(a^ib^jc^{ij})^k \in Z(G)$ for some $k \in \{0, \ldots, p^2-1\}$.

Now by (iii) we have $(a^ib^jc^{ij})^k = a^{ik}b^{jk}c^{ijk}c^{-(1/2)ijk(k-1)}$. Hence $x^k \in Z(G)$ if and only if $a^{ik}b^{jk} \in Z(G)$. By (iv) it follows that $p^2|ik$ and $p^2|jk$. Now $p^2 \nmid i$ and $p^2 \nmid j$. If $p^2 \nmid k$, then we have $p|i$ and $p|j$ so $p^2|ij$ and $p|k$ so $p^2|ik$ and $p^2|jk$. Hence $x^k = (c^{-(1/2)k(k-1)})^{ij} = 1$. If $p^2|k$, then also $x^k = 1$ because k must be 0. It follows that G satisfies Definition (1.1)(a)(b)(c) and so G is a $*$-group. ♦

§ 2. CHARACTER EXTENSION THEORY OF $*$-GROUPS AND DADE'S CORRESPONDENCE IDEA

In his paper [5] Dade deals with a group G with a normal extra special p-subgroup E, whose center $Z(E)$ is central in G and which satisfies certain additional conditions (see also Hypothesis (2.13)) saying that $G/C_G(E/Z(E))$ has a non-trivial normal p'-subgroup acting fixed-point-freely on $E/Z(E)$. It is proved that there exists a complementary subgroup H to the section $E/Z(E)$ of G and a one-to-one correspondence between the irreducible characters of H faithful on $Z(E)$ and the irreducible characters of G faithful on E. The construction of this correspondence requires the existence of a suitable extension $\hat{\chi}$ of a faithful irreducible character χ of E to the semidirect product $H \odot E$. The conditions, mentioned above, enable us to reduce the construction to the case in which H has a normal subgroup K of order coprime to that of E, such that $E = [E, K]$. We are thus led to the situation of Hypothesis (2.1) which describes such semidirect products.

In this section we extend the character theory mentioned above to the case of E being a $*$-group. The procedure of construction is as follows. We show during the proof of Theorem (2.11) that the construction holds unless $E/Z(E)$ is an elementary abelian p-group. A theorem of Wolf asserts then that the construction also holds in that case unless $H \odot E$ contains a subgroup $H \odot F$, where F is an extra special p-group. However, then we are precisely in Dade's construction to be applied on $H \odot F$. It follows then that the construction can be made for $H \odot E$ itself anyway.

Another word on Dade's paper is here in order. If p is odd, Hypothesis (2.1) implies that the extra special p-subgroup E of G is of exponent p and hence is a $*$-group. In this case the above required extension is fairly easy to obtain and in fact we prove the existence of the extension in case that E is any $*$-group of odd order independently, thereby covering Dade's case. If $p = 2$, then Hypothesis (2.1) does not imply that the extra special 2-group E is a $*$-group. So here an original idea had to be found. It was Dade who proved the existence of the required extension which is much harder to establish in this case where p is even. The proof is the bulk of his paper [5]. But see here also Remark (2) (after Theorem (2.12)) for some recent evolvements. It is worth to mention that in his investigations into the structure of minimal non-M-groups, van der Waall used Dade's results for the first time in the literature (see [22] and [23]).

Dade's paper has a great impact in the development of the character theory of (solvable) groups. To mention some recent papers, view also [6]–[9], [14]–[21], [24], [25]. Good survey papers are [1] and [14].

(2.1) HYPOTHESIS. Let G, H, K, and E be groups and p a prime satisfying
(a) E is a normal extra special p-subgroup of G.
(b) H is a complement of E in the semidirect product $G=H\cdot E$.
(c) $K \trianglelefteq H$.
(d) $p \nmid |K|$.
(e) $[E, K]=E$.
(f) $Z(E) \subseteq Z(G)$.

Note that if $p=2$, then Hypothesis (2.1)(f) can be dropped. For $|Z(E)|=2$ in that case and $Z(E)$ char $E \trianglelefteq G$ so $Z(E) \trianglelefteq G$ whence $Z(E)$ is centralized by G. If p is odd it will follow that the next Hypothesis (2.2) is in fact a generalization of Hypothesis (2.1).

(2.2) HYPOTHESIS. Let G, H, K, and E be groups satisfying
(a) E is a normal $*$-group of G of odd order.
(b) H is a complement of E in the semidirect product $G=H\cdot E$.
(c) $K \trianglelefteq H$.
(d) gcd $(|K|, |E|)=1$.
(e) $[E, K]=E$.
(f) $Z(E) \subseteq Z(G)$.

(2.3) PROPOSITION. *Assume* Hypothesis (2.2). *By* Theorem (1.11) *let χ be any of the $\phi(|Z(E)|)$ faithful irreducible characters of E. Then χ is invariant in G.*

PROOF. As χ is defined on E, χ is invariant in E. But $G=H\cdot E$ so let us prove that χ is invariant in H. Let $h \in H$. By (2.2)(a) $\chi^h \in \operatorname{Irr}(E)$ and is also faithful because χ is. Now by Theorem (1.11) any faithful irreducible character of E vanishes off $Z(E)$. And by (2.2)(f) we have $\chi^h|_{Z(E)}=\chi|_{Z(E)}$. It follows that $\chi^h=\chi$. This proves (2.3). ♦

Now remember that the $*$-group E has $\phi(|Z(E)|)$ faithful irreducible characters by Theorem (1.11).

(2.4) THEOREM. *Assume* Hypothesis (2.2). *Let $\chi \in \operatorname{Irr}(E)$ be faithful. Then χ is extendible to an irreducible character $\hat{\chi}$ of G.*

PROOF. We have $C_H(E) \trianglelefteq G$ and thus $EC_H(E)=E \times C_H(E)$. Therefore we can extend χ to an irreducible character $\bar{\chi}:=\chi \otimes 1_{C_H(E)}$ of $E \times C_H(E)$. Note that

$\ker(\bar{\chi}) = C_H(E)$ as χ is faithful. Also by Proposition (2.3) we have $\bar{\chi}$ is invariant in G. In the rest of the proof we work with $\bar{G} := G/C_H(E)$, $\bar{H} := H/C_H(E)$, $\bar{K} := KC_H(E)/C_H(E)$ and $\bar{E} := EC_H(E)/C_H(E)$ and $\bar{\chi}$. One easily checks that the groups $\bar{G}$, $\bar{H}$, $\bar{K}$ and $\bar{E}$ satisfy Hypothesis (2.2) and that $\bar{\chi}$ is faithful on $\bar{E}$ (note that $\bar{E} \simeq E$).

As $C_H(E) \cap E = \{1\}$, we have $C_{\bar{H}}(\bar{E}) = \{\bar{1}\}$. Now we drop the bar notation. Hence we may assume that $C_H(E) = \{1\}$. In words H acts faithfully as automorphism group on E. So we may identify H with its image in Aut (E). Because $Z(E) \subseteq Z(G)$ it follows that $H \subseteq J := C_{\mathrm{Aut}\,(E)}(Z(E)) \trianglelefteq \mathrm{Aut}\,(E)$. By Hypothesis (2.2)(a) $|E|$ is odd, which by Theorem (1.8) is clearly equivalent to $|E/Z(E)|$ is odd. It follows that Corollary (1.3) applies. Write $I = \mathrm{Inn}\,(E)$. We have

(a) $J/I \cong \mathrm{Sp}\,(E)$.

(b) There exists a $\tau \in J$ such that τ inverts $E/Z(E)$, $\tau^2 = 1$, $j = IS$ and $I \cap S = \{1\}$, where $S = C_J(\tau)$.

Now let $\kappa \in K \subseteq H \subseteq J$ say $\kappa = \phi_x \cdot \sigma$ for some $x \in E$ and $\sigma \in S$. Now $x\tau(x) \in Z(E)$. Therefore one easily checks that for all $g \in G$ we have

$$\kappa(\tau\phi_g)\kappa^{-1} = (\phi_x\sigma)(\tau\phi_g)(\sigma^{-1}\phi_{x^{-1}}) = \tau\phi_{\tau(x)\sigma(g)x^{-1}} = \tau\phi_{x^{-1}\sigma(g)x^{-1}}.$$

In particular, if $\sigma = 1_E$, then $\phi_x(\tau\phi_g)\phi_{x^{-1}} = \tau\phi_{x^{-1}gx^{-1}}$. It follows that the group $K.I$ acts by conjugation on the coset τI of J/I. We claim that I acts regularly on the coset τI. We have to show two things:

(i) $C_I(\psi) = \{1\}$ for all $\psi \in \tau I$.

(ii) I acts transitively on τI.

Re (i). Let $\psi \in \tau I$ and write $\psi = \tau\phi_x$ for some $x \in E$. Suppose that $\phi_y\tau\phi_x\phi_{y^{-1}} = \tau\phi_x$ for some $y \in E$. Then $\tau\phi_{\tau(y)xy^{-1}} = \tau\phi_{y^{-1}xy^{-1}} = \tau\phi_x$. It follows that $x^{-1}y^{-1}xy^{-1} \in Z(E)$. But $[x, y] \in Z(E)$ so it follows that $y^2 \in Z(E)$. Hence, as $|E|$ is odd, $y \in Z(E)$ whence $\phi_y = 1_E$. This proves (i).

Re (ii). Let $n = lcm(\exp\,(Z(E)), \exp\,(E/Z(E)))$. Observe, as $Z(E)$ is cyclic, that $\exp\,(Z(E)) = |Z(E)|$. Now define the map $f: E \to Z(E)$ by $f(x) = x^n$ for all $x \in E$. We have the following.

- f is well-defined.

Indeed, let $e = \exp\,(E/Z(E))$, so $n = e \cdot l$ for some $l \in \mathbb{N}$. Hence if $x \in E$, then $x^nZ(E) = x^{el}Z(E) = (x^eZ(E))^l = (1Z(E))^l = Z(E)$. So $x^n \in Z(E)$.

- f is a homomorphism.

First we show that for all $i \in \mathbb{N}$ and for all $x, y \in E$ the following holds:

$$(xy)^i = x^iy^iz^{\frac{1}{2}i(i-1)}, \text{ where } z = [y, x].$$

Indeed, we have $z = y^{-1}x^{-1}yx$ so $yz = x^{-1}yx$ and hence $y^iz^i = x^{-1}y^ix$ because $z \in Z(E)$. Thus $y^ix = xy^iz^i$. Now induction yields

$$\begin{aligned}(xy)^{i+1} &= (xy)^i(xy) = x^iy^iz^{\frac{1}{2}i(i-1)}xy = \\ &= x^iy^ixyz^{\frac{1}{2}i(i-1)} = x^ixy^iz^iyz^{\frac{1}{2}i(i-1)} = \\ &= x^{i+1}y^{i+1}z^{\frac{1}{2}i(i+1)}.\end{aligned}$$

This shows the above formula to be true. Now let $x, y \in E$. Then we have

$$f(xy) = (xy)^n = x^n y^n z^{\frac{1}{2}n(n-1)}.$$

Now let $s = \exp(Z(E))$, so $n = s \cdot m$ for some $m \in \mathbb{N}$. Now n is odd and hence we have

$$z^{\frac{1}{2}n(n-1)} = (z^{\frac{1}{2}(n-1)})^n = (z^{\frac{1}{2}(n-1)})^{sm} = 1.$$

It follows that

$$f(xy) = f(x) \cdot f(y).$$

So f is a homomorphism.

Now let $k \in K$ and $x \in E$. Then

$$[x, k]^n = (x^{-1}x^k)^n = (x^{-1})^n (k^{-1}xk)^n = (x^n)^{-1} k^{-1} x^n k.$$

But $x^n \in Z(E)$, so $[x, k]^n = 1$. But by Hypothesis (2.2)(e), if $x \in E$, then $x = \prod_{i=1}^{t} [x_i, k_i]^{\varepsilon_i}$ with $x_i \in E$, $k_i \in K$ and $\varepsilon_i = \pm 1$. So $x^n = \prod_{i=1}^{t} ([x_i, k_i]^n)^{\varepsilon_i} = 1$. It follows that f is trivial, that is $f(E) = \{1\}$. So in particular we obtain $\exp(E)|n$. On the other hand, as $Z(E)$ is a subgroup of E, $\exp(Z(E))|\exp(E)$. And also $\exp(E/Z(E))|\exp(E)$. So we have $n|\exp(E)$ and hence $n = \exp(E)$. It follows that the $*$-group E satisfying Hypothesis (2.2) has exponent $\exp(E) = lcm(\exp(Z(E)), \exp(E/Z(E)))$.

Now to the transitivity of I on τI. Let $x_1, x_2 \in E$. Then we have to look for an $x \in E$ with $(\tau\phi_{x_1})\phi_x = \phi_x(\tau\phi_{x_2})$. Now this last equation is equivalent with $x_1 x_2^{-1} x^2 \in Z(E)$ (recall that $x\tau(x) \in Z(E)$). Take $x = (x_2)^{-\frac{1}{2}(n-1)} \cdot (x_1)^{\frac{1}{2}(n-1)}$. Then

$$\begin{aligned} x^2 &= (x_2^{-\frac{1}{2}(n-1)} \cdot x_1^{\frac{1}{2}(n-1)})^2 = (x_2^{-\frac{1}{2}(n-1)})^2 \cdot (x_1^{\frac{1}{2}(n-1)})^2 \cdot [x_1^{\frac{1}{2}(n-1)}, x_2^{-\frac{1}{2}(n-1)}] = \\ &= x_2^{-(n-1)} \cdot x_1^{n-1} \cdot z = \\ &= x_2 \cdot x_1^{-1} \cdot z, \text{ as } \exp(E) = n \text{ and } z = [x_1^{\frac{1}{2}(n-1)}, x_2^{-\frac{1}{2}(n-1)}] \in Z(E). \end{aligned}$$

This proves (ii).

The next thing we claim is that $KI = C_{KI}(\tau)I$. For fix a $\kappa \in K$. Then $\kappa\tau\kappa^{-1} = \tau\phi_{x_1}$ for some $x_1 \in E$, as KI acts by conjugation on τI. But by (ii) above I acts transitively on τI, so there exists a $x_2 \in E$ such that $\tau\phi_{x_1} = \phi_{x_2}\tau\phi_{x_2^{-1}}$. It follows that $\phi_{x_2^{-1}} \cdot \kappa \in C_{KI}(\tau)$. So $K \subseteq IC_{KI}(\tau) = C_{KI}(\tau)I$ (recall that $I \trianglelefteq KI$). And this proves the claim.

So $C_{KI}(\tau)$ is a complement in KI to I, as by (i) above $I \cap C_{KI}(\tau) = C_I(\tau) = \{1\}$. But by Hypothesis (2.2)(d) we have $K \cap I = \{1\}$. Hence K is also a complement to I in KI. By the Schur-Zassenhaus Theorem it follows that K is conjugate to $C_{KI}(\tau)$, say $K^{\kappa\phi} = K^{\phi} = C_{KI}(\tau)$ for some $\phi \in I$. So $K = C_{KI}(\phi\tau\phi^{-1})$. Hence replacing τ by $\phi\tau\phi^{-1}$, which does not affect (b) above, we may assume that $K = C_{KI}(\tau)$. Now because I is abelian and gcd $(|K|, |I|) = 1$ we have by Fitting's Lemma that $I = [I, K] \times C_I(K)$. But by Hypothesis (2.2)(e) we have $[I, K] = I$. Hence $C_I(K) = \{1\}$. Now let $\phi \in I$ and suppose that $K \subseteq C_{KI}(\tau)$. Then $\phi \in C_I(K)$ and thus $\phi = 1$. It follows that

τ is the only element of the coset τI centralized by K

(remember $K=C_{KI}(\tau)$). And this forces $N_J(K)\subseteq C_J(\tau)=S$. For if $\psi\in J$, then $\psi=\sigma\phi$ for some $\phi\in I$ and $\sigma\in S$, so $\tau^{\phi}=\tau^{\sigma\phi}=\tau^{\phi}\in I$. Hence if $\psi\in N_J(K)$, then $K=K^{\psi}=C_{KI}(\tau)^{\psi}=C_{KI}(\tau^{\psi})$. So $\tau^{\psi}=\tau$ by the above. This yields $\psi\in C_J(\tau)$. Because $K\trianglelefteq H$ we now have

$$H\subseteq N_J(K)\subseteq C_J(\tau)=S.$$

Now let $\upsilon\in J$ be an involution inverting $E/Z(E)$ and with $K\subseteq C_J(\upsilon)$. So $\upsilon\in N_J(K)$ and hence $\upsilon\in C_J(\tau)$. Now suppose that $\tau\neq\upsilon$. Then order $(\tau\upsilon)=2$. And of course $\tau\upsilon$ acts trivially on $E/Z(E)$. But, as $\tau\upsilon\in J$, $\tau\upsilon$ acts also trivially on $Z(E)$. By Lemma (1.2)(b) it follows that $\tau\upsilon\in I$. But, as $|I|$ is odd, this is a contradiction. Hence $\tau=\upsilon$ and it follows that

there is a unique involution $\tau\in J$ inverting $E/Z(E)$ and with $K\subseteq C_J(\tau)$.

As $H\subseteq S$, it is sufficient to show that χ can be extended in some way to a $\hat{\chi}\in\mathrm{Irr}\ (E\rtimes S)$. Now by Theorem (1.11) there exists a unique faithful $\lambda\in\mathrm{Irr}\ (Z(E))$ such that $[\chi|_{Z(E)},\lambda]\neq 0$ and there exists a $\mu\in\mathrm{Irr}\ (M)$, an extension of λ, such that $\mu^E=\chi$, where M is a maximal abelian normal subgroup of E. Note that χ and λ are fully ramified with respect to $E/Z(E)$, in fact $\chi|_{Z(E)}=$ $=\chi(1)\lambda$ and $\lambda^E=\chi(1)\chi$. Now consider the character triple $(E, Z(E), \lambda)$. As $E/Z(E)$ is abelian and $\lambda(1)=1$ we have $\langle\!\langle x,y\rangle\!\rangle_\lambda=\lambda([x,y])$ for all $x,y\in E$. And if $Z(E)=\langle z\rangle$, then, as $\ker(\lambda)=\{1\}$, we have $\lambda(z)=\zeta$, a primitive $|Z(E)|$-th root of unity, and $\lambda([x,y])=\zeta^i$ if $[x,y]=z^i$. Observe that S preserves the form $\langle\!\langle\ ,\ \rangle\!\rangle_\lambda$ and $Z(E)\rtimes S=Z(E)\times S$ because, as a subgroup of J, S acts trivially on $Z(E)$.

We now show that χ is extendible to $E\rtimes S$. Identify E with $E\rtimes\{1\}$. By Lemma (1.20) it is sufficient to prove that χ is extendible to $E\rtimes Q$ for all Sylow q-subgroups Q of S. If $q\nmid|E|$, then by Satz V.17.12 of [11] χ can be extended to $E\rtimes Q$, as gcd $(\chi(1),|Q|)=1$. So assume $q\,\big|\,|E|$. Now by Theorem (1.8) we can write $E=P_1\times P_2\times\cdots\times P_t$, where $P_1\in\mathrm{Syl}_q(E)$ and $P_i\in\mathrm{Syl}_{p_i}(E)$ $(2\leq i\leq t)$ and $\{q,p_2,\ldots,p_t\}$ are the different prime divisors of $|E|$. The groups $P_1,\ldots,P_t$ are again $*$-groups. It follows then by Theorem (1.11) that there exist $\chi_i\in\mathrm{Irr}\ (P_i)$ $(1\leq i\leq t)$ which are faithful, lie over a unique faithful $\lambda_i\in\mathrm{Irr}\ (Z(P_i))$ and such that $\chi=\chi_1\otimes\chi_2\otimes\cdots\otimes\chi_t$ and $\lambda=\lambda_1\otimes\lambda_2\otimes\cdots\otimes\lambda_t$. Now again by ([11], Satz V.17.12) we can extend χ_i to $\hat{\chi}_i\in\mathrm{Irr}\ (P_i\rtimes Q)$ for all $i\in\{2,\ldots,t\}$, where we have fixed a Sylow q-subgroup Q of S.

Now consider χ_1. Now $Z(P_1)\subseteq Z(E)\subseteq Z(G)$, so $Q\subseteq J$ acts trivially on $Z(P_1)$. Also observe that if $x,y\in E$, say $x=(x_1,\ldots,x_t)$ and $y=(y_1,\ldots,y_t)$ with $x_i,y_i\in P_i$ $(1\leq i\leq t)$, then we have

$$\begin{aligned}\langle\!\langle x,y\rangle\!\rangle_\lambda&=\langle\!\langle x_1,y_1\rangle\!\rangle_{\lambda_1}\cdot\ldots\cdot\langle\!\langle x_t,y_t\rangle\!\rangle_{\lambda_t}=\\&=\lambda_1([x_1,y_1])\cdot\ldots\cdot\lambda_t([x_t,y_t]).\end{aligned}$$

It follows that the form $\langle\!\langle\ ,\ \rangle\!\rangle_{\lambda_1}$ is trivial on $Z(P_1)$. Therefore we can choose $A\supseteq Z(P_1)$, with A a maximal Q-invariant subgroup of P_1 on which the form $\langle\!\langle\ ,\ \rangle\!\rangle_{\lambda_1}$ is trivial. As $P_1'\subseteq Z(P_1)\subseteq A$, we have $A\trianglelefteq P_1$. Now let $B=A^{\perp}$, computed in $(P_1, Z(P_1),\lambda_1)$. Suppose that $B^{\perp}\supsetneq A$. Because P_1 is a q-group we can choose a Q-invariant subgroup A_0 with $B \supseteq A_0 \supseteq A$, $|A_0 : A| = q$. So $A_0 = \langle A, x \rangle$ for some $x \in A_0 - A$ with $x^q \in A$. Now let $a_0, b_0 \in A_0$, say $a_0 = ax^i$ and $b_0 = bx^j$ for some $a, b \in A$ and $i, j \in \mathbb{Z}$. Then

$$\langle\!\langle a_0, b_0 \rangle\!\rangle_{\lambda_1} = \langle\!\langle a, b \rangle\!\rangle_{\lambda_1} \cdot \langle\!\langle a, x^j \rangle\!\rangle_{\lambda_1} \cdot \langle\!\langle x^i, b \rangle\!\rangle_{\lambda_1} \cdot \langle\!\langle x, x \rangle\!\rangle_{\lambda_1}{}^{ij} = 1$$

by CTP (a), (b), (d) and the fact that $x^i, x^j \in B$ and $\langle\!\langle\ ,\ \rangle\!\rangle_{\lambda_1}$ is trivial on A. It follows that $\langle\!\langle\ ,\ \rangle\!\rangle_{\lambda_1}$ is trivial on A_0, which contradicts the choice of A. Hence $B = A^\perp = A$. Now because λ_1 is faithful we have

$$\begin{aligned} P_1^\perp &= \{x \in P_1 | \langle\!\langle x, y \rangle\!\rangle_{\lambda_1} = 1 \text{ for all } y \in P_1\} = \\ &= \{x \in P_1 | \lambda_1([x, y]) = 1 \text{ for all } y \in P_1\} = \\ &= \{x \in P_1 | [x, y] = 1 \text{ for all } y \in P_1\} = Z(P_1). \end{aligned}$$

Hence CTP (j) yields $|A : Z(P_1)| = |P_1 : A|$. Also, because $\langle\!\langle\ ,\ \rangle\!\rangle_{\lambda_1}$ is trivial on A it follows by a similar argument as above that, $A = A^\perp = Z(A)$. Hence A is a maximal abelian normal subgroup of P_1 by Theorem (1.11). Further, let $x \in P_1$ and $a \in A$ with $\tau(a) = a$. Now $x\tau(x) = z$ for some $z \in P_1$, as P_1 char E (so τ acts on P_1) and as E is nilpotent. By CTP (g) we have

$$\begin{aligned} \lambda_1([x, a]) &= \langle\!\langle x, a \rangle\!\rangle_{\lambda_1} = \langle\!\langle x, \tau(a) \rangle\!\rangle_{\lambda_1} = \langle\!\langle \tau(x), a \rangle\!\rangle_{\lambda_1} = \langle\!\langle x^{-1}z, a \rangle\!\rangle_{\lambda_1} \\ &= \langle\!\langle x^{-1}, a \rangle\!\rangle_{\lambda_1} \cdot \langle\!\langle z, a \rangle\!\rangle_{\lambda_1} = \langle\!\langle x^{-1}, a \rangle\!\rangle_{\lambda_1}. \end{aligned}$$

So $\lambda_1([x, a]^2) = 1$, so because λ_1 is faithful we have $[x, a]^2 = 1$ and hence, as $|E|$ is odd, we have $[x, a] = 1$. It follows that $C_A(\tau) = Z(P_1)$. Now $\langle \tau \rangle$ acts on A and $|A|$ is odd so by Fitting's Lemma we have $A = [A, \tau] \times Z(P_1)$. Because A is Q-invariant and $Q \subseteq S = C_J(\tau)$ we have $Q \subseteq N_J([A, \tau])$. We conclude that $A \rtimes Q = ([A, \tau] \rtimes Q) \times Z(P_1)$. And thus λ_1 is extendible to $\upsilon \in \mathrm{Irr}\ (A \rtimes Q)$ by taking $\upsilon = 1 \otimes \lambda_1$. Now by Mackey's Theorem $(\upsilon^{P_1 \rtimes Q})|_{P_1} = \mu^{P_1}$, where $\mu = \upsilon|_A$. But A is a maximal abelian normal subgroup of P_1 and $(\upsilon|_A)^{P_1}(1) = |P_1 : A| \upsilon(1) = |P_1 : A|$ and $\upsilon|_{Z(P_1)} = \mu|_{Z(P_1)} = \lambda_1$. So by Theorem (1.11) we obtain $\chi_1 = \mu^{P_1} = (\upsilon^{P_1 \rtimes Q})|_{P_1}$. So if we take $\hat{\chi}_1 := \upsilon^{P_1 \rtimes Q}$, then $\hat{\chi}_1$ is an extension of χ_1 to $P_1 \rtimes Q$.

We have now proved that each χ_i can be extended to a $\hat{\chi}_1 \in \mathrm{Irr}\ (P_1 \rtimes Q)$. Now let $\mathfrak{X}_i$ be representations of $P_i \rtimes Q$, which afford $\hat{\chi}_i$. Now let $g \in E \rtimes Q$, say $g = xy$ with $x = (x_1, \ldots, x_t) \in E$, where $x_i \in P_i$ $(1 \leq i \leq t)$ and $y \in Q$. Define

$$\mathfrak{X}(g) = \mathfrak{X}_1(x_1 y) \otimes \mathfrak{X}_2(x_2 y) \otimes \cdots \otimes \mathfrak{X}_t(x_t y).$$

Then one easily checks that $\mathfrak{X}$ is a well defined representation of $E \rtimes Q$ of degree $\prod_{i=1}^t \hat{\chi}_i(1) = \prod_{i=1}^t \chi_i(1) = \chi(1)$. Let $\hat{\chi}$ be the character afforded by $\mathfrak{X}$. Note that $(\text{trace } \mathfrak{X})|_{Z(E)} = \chi(1)\lambda$ and by Frobenius reciprocity and Theorem (1.11) it follows that $\hat{\chi}|_E = \chi$. So indeed χ can be extended to a $\hat{\chi} \in \mathrm{Irr}\ (E \rtimes Q)$. This proves (2.4). ♦

(2.5) THEOREM. *Assume* Hypothesis (2.1). *So E has $p-1$ faithful irreducible characters. Let χ be one of them. Then χ is extendible to an irreducible character $\hat{\chi}$ of G.*

PROOF. If $2\nmid |E|$, then Hypothesis (2.1) implies that exp $(E)=p$ (see for instance inside the proof of Theorem (2.4) under Re (ii)) and hence E is a $*$-group. So in this case we can apply Theorem (2.4). If $2 \mid |E|$ then see Theorem (1.2) in [5]. This proves (2.5). ♦

REMARK. We have just established that Theorem (2.5) can be seen as a corollary to Theorem (2.4) in case $p \neq 2$. But for $p=2$, Dade's Theorem (1.2) in [5] (which is Theorem (2.5)) has its "own" proof, as given in [5]. We come back to this later after the proof of Theorem (2.12), as already suggested in the introduction to this section.

(2.6) HYPOTHESIS. Let G, H, L and E be groups and p a prime satisfying
(a) E is a normal subgroup of G.
(b) H is a complement of E in the semidirect product $G=H \cdot E$.
(c) $L/C_H(E) \trianglelefteq H/C_H(E)$.
(d) $E/Z(E)$ is a non-trivial elementary abelian p-group.
(e) $p \nmid |L/C_H(E)|$.
(f) $[E/Z(E), L]=E/Z(E)$.
(g) $Z(E) \subseteq Z(G)$.
(h) $Z(E)$ is cyclic.

We derive some properties of the group E in Hypothesis (2.6).

(2.7) PROPOSITION. *Assume* Hypothesis (2.6). *Then the following hold.*
(a) $E=P \times C_n$, *where P is a non-abelian p-group, $p \nmid n$, $[P, P] \subseteq \Phi(P) \subseteq Z(P)$ and $Z(P)$ is cyclic.*
(b) $|[P, P]|=p=|[E, E]|$.
(c) *Every non-linear irreducible character of P is faithful and has degree $\sqrt{|P/Z(P)|}$.*
(d) *If $\tau \in \mathrm{Aut}\,(E)$ is trivial on $Z(E)$ and on $E/Z(E)$, then $\tau \in \mathrm{Inn}\,(E)$.*

PROOF. (a). This follows from the nilpotency of E, Hypothesis (2.6)(d) and Hypothesis (2.6)(h).

(b). Now $P' \subseteq Z(P)$ by (a). So if $x, y \in P$, then $[x, y]^p=[x, y^p]$ and $y^p \in Z(P)$ by Hypothesis (2.6)(d), so $[x, y]^p=1$. Hence P', being cyclic and non-trivial, has order p. By (a) $E'=P' \times \{1\}$.

(c). Let $\chi \in \mathrm{Irr}\,(P)$ with $\chi(1) \neq 1$. Then because $|P'|=p$ and $P' \subseteq Z(P)$ we have by Exercise (2.13) in [13] that $\chi(1)^2=|P/Z(P)|$. Now by the nilpotency of P we have $\ker(\chi)=\{1\}$ if and only if $\ker(\chi) \cap Z(P)=\{1\}$. As $Z(P)$ is cyclic and $|P'|=p$, we see that P' is the unique subgroup of $Z(P)$ of order p. Hence it follows that $\ker(\chi) \supsetneq \{1\}$ if and only if $\ker(\chi) \cap Z(P) \supsetneq \{1\}$ if and only if $\ker(\chi) \supseteq P'$ if and only if $P/\ker(\chi)$ is abelian. But by Corollary (2.30) in [13] we have $\chi(1)^2 \leq |P/Z(\chi)|$ and this yields $Z(\chi)=Z(P)$. So if $\ker(\chi) \supsetneq \{1\}$, then $Z(P)/\ker(\chi)=Z(\chi)/\ker(\chi)=Z(P/\ker(\chi))=P/\ker(\chi)$. Hence $P=Z(P)$ which contradicts P being non-abelian.

(d). Clearly Inn (E) is trivial on $Z(E)$ and on $E/Z(E)$. Now by Hypothesis (2.6)(d) write $E/Z(E)=\langle\bar{x}_1\rangle\times\cdots\times\langle\bar{x}_t\rangle$ with $x_i^p\in Z(E)$ and $x_i\in E-Z(E)$ for all i.

Let $\tau\in\operatorname{Aut}(E)$ be trivial on $E/Z(E)$ and on $Z(E)$. Now $E=\langle x_1,\dots,x_t\rangle Z(E)$ and τ is specified by the images $\tau(x_i)$ $(1\le i\le t)$. So $\tau(x_i)=x_iz_i$ for some $z_i\in Z(E)$. But $x_i^p\in Z(E)$, so $\tau(x_i^p)=x_i^p=(x_iz_i)^p=x_i^pz_i^p$ whence $z_i^p=1$ and z_i lies in the unique subgroup $P'\times\{1\}$ of $Z(E)$ of order p. It follows that there are at most p possible choices for the z_i's and hence the number of possible τ is $\le\prod_{i=1}^t p=p^t=|E/Z(E)|$. This proves (2.7). ♦

REMARKS. (i) Compare Proposition (2.7)(d) with Lemma (1.2)(b).

(ii) It follows by Proposition (2.7)(a) that the group P is a p-group with $Z(\Phi(P))$ cyclic. All the p-groups with this property have been classified by Berger, Kovács and Newman ([2]). So this classification yields a classification of all the groups E satisfying Hypothesis (2.6).

By Proposition (1.5) the group E in Hypothesis (2.6) has indeed faithful irreducible characters. The next theorem is due to Th. Wolf. We mention also the proof, in view of Remark (2) after the proof of Theorem (2.12).

(2.8) THEOREM. (Wolf [27]) *Assume* Hypothesis (2.6). *Let χ be a faithful irreducible character of E. Then χ is extendible to an irreducible character $\hat{\chi}$ of G.*

PROOF. As in the proof of Theorem (2.4) we may assume that $C_H(E)=\{1\}$. So by Hypothesis (2.6)(c), (e) we have $L\trianglelefteq H$ and $p\nmid|L|$. If $E'=Z(E)$, then by Proposition (2.7)(a), (b) E is an extra special p-group. Hypothesis (2.6)(f) yields $[E,L]Z(E)=E$. But $\Phi(E)=Z(E)$ so $E=[E,L]$ in this case. It follows that Hypothesis (2.6) is equivalent to Hypothesis (2.1) in this case and hence Theorem (2.5) applies. So let us assume that $E'\subsetneq Z(E)$.

Now by Proposition (2.7)(a) $E=PN$ with P, N char E, $N\cap P=\{1\}$, N cyclic, P a p-group, $p\nmid|N|$ and $N\subseteq Z(E)$. Now Hypothesis (2.6)(f) yields $[E,L]Z(E)==E$. Since $N\subseteq Z(G)$ by Hypothesis (2.6)(g) we have $[E,L]=[P,L]$ and $Z(E)==Z(P)N=Z(P)\times N$, hence $E=[E,L]Z(E)=[P,L]Z(P)N$. It follows that $P=[P,L]Z(P)$ (*).

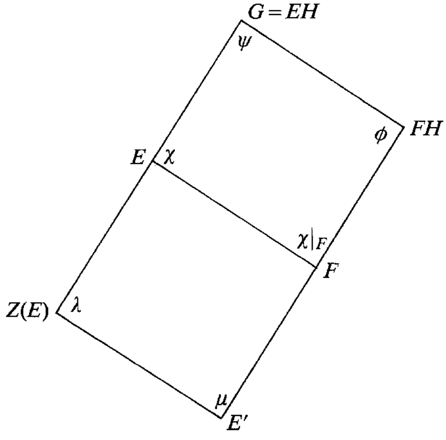
By Fitting's Lemma this implies $C_{P/Z(P)}(L)=\{1\}$, so $C_P(L)\subseteq Z(P)$. But, as $Z(P)\subseteq Z(E)\subseteq Z(G)$, $C_P(L)\supseteq Z(P)$, whence $C_P(L)=Z(P)$. So $C_{P/P'}(L)==Z(P)/P'$. Also $C_E(L)=C_P(L)N=Z(P)N=Z(E)$. Now $C_{E/E'}(L)==C_{P/P'}(L)(NP'/P')=(Z(P)P'N)/P'=(Z(P)N)/P'=Z(E)/E'$, because $E'=P'$ by Proposition (2.7)(a). Moreover, $[E,L]E'=[P,L]P'$. So then we have by Fitting's Lemma

$$E/E'=(P/P')(NP/P')=([P/P',L]\times C_{P/P'}(L))\cdot NP'/P'=$$

$$=[E/E',L]\times C_{E/E'}(L)=F/E'\times Z(E)/E' \quad (**)$$

where $F=[E,L]E'=[P,L]P'$. Note that by (*) $P'=[P,L]'\subseteq[P,L]$, so $F=[P,L]$ is a p-group and $F\trianglelefteq G$, because $L\trianglelefteq H$. Further $C_p\simeq E'=P'=F'$. By (**)

we see that $E/Z(E) \simeq P/Z(P) \simeq F/F'$, which is elementary abelian of exponent p. As $E = FZ(E)$, we have $Z(F) \subseteq Z(E)$. But Hypothesis (2.6)(g) yields $Z(E) \cap F \subseteq Z(F)$, hence $Z(F) = F \cap Z(E) = [P, L] \cap Z(P)N = [P, L] \cap Z(P)$. Now from (**) it follows that $[P, L] \cap Z(P) \subseteq P'$. Now $F \trianglelefteq P$ and $F \supsetneqq \{1\}$, so $F \cap Z(P) \supsetneqq \{1\}$, whence $[P, L] \cap Z(P) = P'$. Thus $Z(F) = P' = F' = E'$. Observe that $F' \subseteq \Phi(F) = F^p F' \subseteq Z(F)$ by a remark above. So $F' = \Phi(F) = Z(F)$ and we have proved that F is an extra special p-group.



Note that $F \subsetneqq E$, as $E' \subsetneqq Z(E)$. Let $\chi|_{Z(E)} = \chi(1)\lambda$ with $\lambda \in \operatorname{Irr}(Z(E))$. Now λ is invariant in G and so in E by Hypothesis (2.6)(g). Also $\mu := \lambda|_{E'} \in \operatorname{Irr}(E')$. As E is a central product of F and $Z(E)$ we have $\chi|_F \in \operatorname{Irr}(F)$. The group F satisfies Hypothesis (2.6) (or equivalently Hypothesis (2.1)) and so by induction on $|G|$ $\chi|_F$ extends to some $\phi \in \operatorname{Irr}(FH)$. Note that $\chi|_F$ is faithful and hence vanishes off $Z(F)$. But $Z(F) = E' \subseteq Z(E) \subseteq Z(G)$. It follows that $\chi|_F$ is invariant in H. Also λ is invariant in H. Now we claim that χ is invariant in G. It suffices to show that χ is invariant in H, as $G = HE$. Fix a $h \in H$. Then $(\chi^h)|_F = (\chi|_F)^h = \chi|_F$. It follows by Corollary (6.17) of [13] that we have $\chi^h = \beta\chi$ for some linear $\beta \in \operatorname{Irr}(E/F)$. This implies $\lambda^h = \beta\lambda$, so $\lambda = \beta\lambda$ and hence $\beta|_{Z(E)} = 1_{Z(E)}$. But $E = FZ(E)$ and $\ker(\beta) \supseteq F$, so $\beta = 1_E$, which proves the claim. It is easy to see that $\phi = \psi|_{FH}$ for some $\psi \in \operatorname{Irr}(G|\chi)$. Moreover, $\psi|_{FH}(1) = \psi(1) = \phi(1) = \chi|_F(1) = \chi(1)$ and $[\psi|_E, \chi] \neq 0$, hence $\psi|_E = \chi$. This proves (2.8). ♦

We are now in position to generalize Hypothesis (2.2) and prove the existence of the desired extension thereof. In fact we have somewhat more.

(2.9) HYPOTHESIS. Let G, H, K and E be groups satisfying

(a) E is a normal $*$-group of G.

(b) H is a complement of E in the semidirect product $G = H \cdot E$.

(c) $K \trianglelefteq H$.

(d) $\gcd(|K|, |E|) = 1$.

(e) $[E, K]Z(E) = E$.

(f) $Z(E) \subseteq Z(G)$.

REMARK. Hypothesis (2.2) implies Hypothesis (2.9) if E is of odd order. Notice the difference between the Hypotheses (2.2)(e) and (2.9)(e). Observe that Proposition (2.3) is still true under the assumptions of Hypothesis (2.9).

(2.10) LEMMA. *Assume* Hypothesis (2.9). *Let* $Z(E) \subseteq F \subseteq E$ *with* F K*-invariant. Then* $[F, K]Z(E) = F$. *Moreover if* F *is abelian, then* $F = [F, K] \times Z(E)$.

PROOF. Hypothesis (2.9)(e) is equivalent to $[E/Z(E), K] = E/Z(E)$. So by Fitting's Lemma it follows that $C_{E/Z(E)}(K) = \{1\}$. Since gcd $(|K|, |E|) = 1$, we have $C_{E/Z(E)}(K) = C_E(K)Z(E)/Z(E)$. Hence $C_F(K) = C_E(K) \cap F \subseteq Z(E) \cap F = Z(E)$, as $Z(E) \subseteq F$. On the other hand Hypothesis (2.9)(f) implies that $Z(E) \cap F = Z(E) \subseteq C_F(K)$. So now we have $Z(E) = C_F(K)$. Again by gcd $(|K|, |E|) = 1$ we have $F = [F, K]C_F(K)$. So $F = [F, L]C_F(K) = [F, L]Z(E)$. If F is abelian, then the desired result follows from Fitting's Lemma. This proves (2.10). ♦

Assume Hypothesis (2.9). If E is non-abelian then there are faithful $\chi \in \mathrm{Irr}\,(E)$ by Theorem (1.11). If E is abelian, then it follows from Definition (1.1) that E is cyclic. So there are faithful (linear) $\chi \in \mathrm{Irr}\,(E)$ too.

(2.11) THEOREM. *Assume* Hypothesis (2.9). *Then each faithful irreducible character* χ *of* E *is extendible to an irreducible character* $\hat{\chi}$ *of* G.

PROOF. By the tensor-product argument used in the proof of Theorem (2.4) we may assume that E is a non-abelian p-group. (Note that by Satz V.17.12 of [11] any inert linear character of E can always be extended to G; remember Theorem (1.8)). Now let G be a counterexample of minimal order. As in the proof of Theorem (2.4) we may assume that $C_H(E) = \{1\}$. In this counterexample of minimal order we claim that $E/Z(E)$ is an elementary abelian p-group.

For suppose $E/Z(E)$ is not elementary abelian. Then we can choose $E \supsetneqq D \supsetneqq Z(E)$ with $D/Z(E)$ char $E/Z(E)$. But $Z(E)$ char E, hence D char E. But $E \trianglelefteq G$, hence $D \trianglelefteq G$. Now let $D \supseteq F \supsetneqq Z(E)$ be such that $F/Z(E)$ is a non-trivial chief factor of G. We distinguish two cases.

CASE (a). F is non-abelian.

Then $Z(F) \subsetneqq F$. But as $Z(E) \subseteq Z(G)$ and $Z(E) \subseteq F$, we have $Z(E) \subseteq Z(F)$. Also $Z(F)$ char $F \trianglelefteq G$, so $Z(F) \trianglelefteq G$. Because $F/Z(E)$ is a chief factor of G we now have $Z(E) = Z(F)$. Now by Theorem (1.14) we have that F is also a $*$-group and $E = FC_E(F)$ with $F \cap C_E(F) = Z(F) = Z(E) = Z(C_E(F))$. Note that $C_E(F) \trianglelefteq G$, as $F \trianglelefteq G$. Now $|F| < |E|$ because $D \subsetneqq E$. And as E and F are non-abelian groups it follows that $C_E(F)$ is also non-abelian $*$-group with $|C_E(F)| < |E|$ (observe that $|F| \cdot |C_E(F)|/|Z(F)| = |E|$). Now it follows by Lemma (2.10) that the groups HF, H, K and F and the groups $HC_E(F)$, H, K and $C_E(F)$ satisfy Hypotheses (2.9). Therefore they are no counterexamples to the theorem.

Following Theorem (1.11) let χ lie over a unique $\lambda \in \mathrm{Irr}\,(Z(E))$ with $\ker(\lambda) = \{1\}$ and $\lambda(1)=1$. By the fact that $Z(E)=Z(F)=Z(C_E(F))$ there exist by Theorem (1.11) again a unique faithful $\phi \in \mathrm{Irr}\,(F)$ and $\psi \in \mathrm{Irr}\,(C_E(F))$, which lie over λ and such that $[\chi|_F, \phi] \neq 0 \neq [\chi|_{C_E(F)}, \psi]$. Also $\phi(1)^2 = |F/Z(F)|$ and $\psi(1)^2 = |C_E(F)/Z(C_E(F))|$ and $\chi(1)^2 = |E/Z(E)|$ so $\chi(1) = \phi(1)\cdot\psi(1)$. By induction there exist $\hat{\phi} \in \mathrm{Irr}\,(HF)$ and $\hat{\psi} \in \mathrm{Irr}\,(HC_E(F))$ such that $\hat{\phi}|_F = \phi$ and $\hat{\psi}|C_E(F) = \psi$. Now let $\mathfrak{Y}$ respectively $\mathfrak{Z}$ be representations of HF respectively $HC_E(F)$ such that $\mathfrak{Y}$ affords $\hat{\phi}$ and $\mathfrak{Z}$ affords $\hat{\psi}$. Now $G = HE = HFC_E(F)$. Let $g \in G$, say $g = hxc$ for some $h \in H$, $x \in F$ and $c \in C_E(F)$. Define

$$\mathfrak{X}(g) = \mathfrak{Y}(hx) \otimes \mathfrak{Z}(hc).$$

Then one easily checks that $\mathfrak{X}$ is a well defined representation of $G = H\cdot E$ of degree $\phi(1)\cdot\psi(1) = \chi(1)$. Let $\hat{\chi}$ be the character afforded by $\mathfrak{X}$. Note that $(\mathrm{trace}\ \mathfrak{X})|_{Z(E)} = \chi(1)\lambda$ and by Frobenius reciprocity and Theorem (1.11) it follows that $\hat{\chi}|_E = \chi$. So indeed χ can be extended to a $\hat{\chi} \in \mathrm{Irr}\,(G)$. This contradicts the choice of G.

CASE (b). F is abelian.

Let M be a maximal abelian H-invariant subgroup of E with $M \supseteq F$. Then we have by Lemma (2.10) $M = [M, K] \times Z(E)$. Define $\mu = 1 \otimes \lambda \in \mathrm{Irr}\,(M)$. Then $\ker(\mu) = [M, K] =: N$ as λ is faithful. Note that $M = NZ(E)$ and $N \cap Z(E) = \{1\}$. Also μ is H-fixed. Now let $T = I_E(\mu)$. Then T is H-invariant and $T \supseteq M \supseteq F$. Again we distinguish two cases.

CASE (b)(i). $T \supsetneqq M$.

Now T is H-invariant so T is not abelian. For if this would be the case, then $T = E$ by the choice of M and hence μ would be invariant in E then. But then $\chi|_M = \chi(1)\mu$. But χ vanishes off $Z(E)$ and it would follow that $M = F = Z(E)$ which is absurd.

Of course $N \trianglelefteq T$. Now let $\bar{x} \in Z(T/N)$, so $[x, y] \in \ker(\mu)$ for all $y \in T$. But $[x, y] \in E' \cap \ker(\mu) \subseteq Z(E) \cap \ker(\mu) = \ker(\lambda) = \{1\}$. So $x \in Z(T)$ and we have $Z(T/N) \subseteq Z(T)N/N$. Now $M \subseteq MZ(T) \subseteq T$ and $MZ(T)$ is abelian and H-invariant. So by the choice of M and the fact that T is not abelian we have $M \supseteq Z(T)$. Hence $Z(T)N/N \subseteq MN/N = M/N = Z(E)N/N \subseteq Z(T/N)$, where the last inclusion holds by Hypothesis (2.9)(f). It follows that $Z(T)N = M$. Now, as $N \trianglelefteq T$ we have $[T, N] \subseteq N \cap E' \subseteq N \cap Z(E) = \{1\}$. So $N \subseteq Z(T)$. Hence it follows that $M = Z(T) = NZ(E)$.

Also $T' \subseteq E' \subseteq Z(E) \subseteq Z(G) \cap T \subseteq Z(T) \subseteq E$. By Lemma (2.10) we have $T = [T, K]Z(E)$ so $T = [T, K]Z(T)$. And this is equivalent to $[T/Z(T), K] = T/Z(T)$.

Notice again that N is HF-invariant. Write the bar notation for reckoning in HT modulo N. We have found above that $\bar{T} = [\bar{T}, \bar{K}]\overline{Z(T)} = [\bar{T}, \bar{K}]Z(\bar{T})$. Let $T \supseteq X_1 \supsetneqq M$ be such that $\overline{X_1}$ is a minimal $\bar{H}$-invariant subgroup of $\bar{T}$, properly containing $\bar{M}$. As $Z(E) \cap N = \{1\}$, it follows that $\overline{X_1}$ is not abelian; indeed, X_1 is isomorphic to a subgroup of $X_1/Z(E) \times \overline{X_1}$ and $X_1/Z(E)$ is abelian by

$X_1 \subseteq T' \subseteq E' \subseteq Z(E)$, and, as X_1 is H-invariant, the choice of M yields $\overline{X_1}$ not abelian. Now $Z(X_1)$ is an H-invariant abelian subgroup of T. Hence $\overline{Z(X_1)} = Z(\overline{X_1}) = Z(\bar{T}) = \bar{M}$ is cyclic and X_1/M is an elementary abelian chief section of HT. Since now any automorphism of $\overline{X_1}$ which centralizes both $\overline{X_1}/Z(\overline{X_1})$ and $Z(\overline{X_1})$ is inner (by the proof of (2.7, d)) it follows easily that $\bar{T} = \overline{X_1} C_{\bar{T}} \overline{X_1}$. The group $C_{\bar{T}}(\overline{X_1})$ is $\bar{H}$-invariant too, and by the choice of M, either $C_{\bar{T}}(\overline{X_1}) = \{1\}$ or $C_{\bar{T}}(\overline{X_1})$ is not abelian. In the latter case we split further into minimal $\bar{H}$-invariant subgroups. Therefore

$$\bar{T} = \overline{X_1}\,\overline{X_2} \ldots \overline{X_t}, \overline{X_i} \cap \overline{X_j} = Z(\overline{X_i}) = Z(\overline{X_j}) = Z(\bar{T}), \text{ for all } i, j,$$

and any $\overline{X_i}$ is minimal $\bar{H}$-invariant and not abelian (by the assumption on M) and $\overline{X_i}/Z(\overline{X_i})$ is elementary abelian. Especially $\bar{T}/Z(\bar{T})$ is elementary abelian.

Now theorem (1.11) yields $\mu^{\bar{T}} = \sqrt{|\bar{T}|}\psi$, with $\psi \in \mathrm{Irr}\ (\bar{T})$, $\psi_{Z(\bar{T})} = \sqrt{|\bar{T}|}\mu$. Thus $I_{\overline{HT}}(\psi) = \overline{HT}$, just as μ is HT-invariant. Hence $\overline{HT}$ satisfies the Hypothesis (2.6) with the appropriate change of symbols, viz.

$$\{G, H, L, E\} \rightarrow \{\overline{HT}, \bar{H}, \bar{K}C_{\bar{H}}(\bar{T}), \bar{T}\}.$$

The character ψ is faithful on $\bar{T}$. Hence by Theorem (2.8), applied on the group $\overline{HT}$, ψ has an extension $\hat{\psi} \in \mathrm{Irr}\ (HT)$ with ker $(\hat{\psi}) \supseteq N$ and with $\hat{\psi}|_T|_{Z(E)} = \psi_{Z(E)} \ni \lambda$. Now, $T \lhd E$ by $T' \subseteq E' \subseteq Z(E) \subset M \subset T \subseteq E$ and $HT \cap E = T$, as HE is a semi-direct product of H with E. Hence Mackey's Theorem gives the result $\hat{\psi}|_T|^E = (\hat{\psi}^{HTE})_E = \hat{\psi}^{HE}|_E$. Since $T = I_E(\mu)$ and $\psi_M \ni \mu$, $\hat{\psi}|_T|^E = \psi^E$ is irreducible. Thus $\hat{\psi}^{HTE}$ is irreducible and $\hat{\psi}^{HTE} = \hat{\psi}^{HE}$ extends ψ^E. Hence, as soon as we have proved that ψ^E is faithful on E, we have $\psi^E = \chi$ by Theorem (1.11), contradicting the choice of G. Now indeed, ker $(\psi^E) \lhd E$, so, if ker $(\psi^E) \neq \{1\}$, then the nilpotency of E gives ker $(\psi^E) \cap Z(E) \neq \{1\}$. Now notice, that $\psi^E(1)\lambda = \psi^E|_{Z(E)}$ and $\lambda(a) = 1$ if and only if $a = 1$, just by the faithfulness of λ on $Z(E)$. This yields that ker $(\psi^E) \cap Z(E)$ is in fact equal to $\{1\}$. Therefore indeed ker $(\psi^E) = \{1\}$ and so $\psi^E = \chi$ with $\hat{\psi}^{HE} \in \mathrm{Irr}\ (HE)$ and $\hat{\psi}^{HE}|_E = \chi$. Hence we have a contradiction to the choice of G.

CASE (b)(ii). $T = M$.

We have $I_E(\mu) = M$. So by Theorem (6.11) of [13] and the fact $[\mu|_{Z(E)}, \lambda] \neq 0$ we have $\mu^E = \chi$. Now since $\mu(1) = 1$ and HM is semidirect, Satz V.17.12 of [11] asserts the existence of an extension $\hat{\mu} \in \mathrm{Irr}\ (HM)$ of μ. But then, as $HM \cap E = (H \cap E)M = M$ it follows by Mackey's Theorem that $(\hat{\mu}|_M)^E = \chi = (\hat{\mu}^{HE})|_E$. So it follows that $\hat{\mu}^{HE}$ extends χ. This contradicts the choice of G.

We now have proved the above claim. So $E/Z(E)$ is an elementary abelian p-group. But now we have arrived just in the Hypothesis (2.6). Hence Theorem (2.8) applies and obviously G is not a counterexample. This proves (2.11). ♦

(2.12) THEOREM. *Let G, H, K and E be groups satisfying*

(a) *H is a complement of E in the semidirect product $G = H \cdot E$,*

(b) *$[E, E]$ has prime order p and $E/Z(E)$ is an elementary abelian p-group,*

(c) *$Z(E) \subseteq Z(G)$,*

(d) $K \trianglelefteq H$ *and* $\gcd(|E|, |K|) = 1$,

(e) $[E, K]Z(E) = E$.

Then any non-linear irreducible character of E can be extended to a character of G.

PROOF. (1) Suppose $\chi \in \mathrm{Irr}(E)$ is not faithful. Then, as E is nilpotent, $\ker(\chi) \cap Z(E) \neq \{1\}$. So there is $S \subseteq \ker(\chi) \cap Z(E)$, S being of prime order. As $\chi(1) \neq 1$, $S \cap E' = \{1\}$. Therefore E/S is not abelian. Note that now $Z(E)/S = Z(E/S)$. It holds that $S \subseteq Z(E) \subseteq Z(G)$ and so $G/S = HS/S \cdot E/S$, $|(E/S)'| = p$, $E/S/Z(E/S)$ is an elementary abelian p-group,

$$Z(E/S) = Z(E)/S \subseteq Z(G)/S \subseteq Z(G)/S,$$

$KS/S \trianglelefteq HS/S$, $\gcd(|KS/S|, |E/S|) = 1$ and $[E/S, KS/S]Z(E/S) = E/S$. Hence we can apply induction. It follows that there is a $\hat{\chi} \in \mathrm{Irr}(G/S)$, extending χ if viewed as character of E/S. But then, of course, if $\hat{\chi}$ is regarded as irreducible character of G, it extends χ.

(2) Let $\chi \in \mathrm{Irr}(E)$ be faithful. Hence $Z(E)$ is cyclic. Since $E' \subseteq Z(E)$, it follows that $\chi(1)^2 = |E/Z(E)|$, by Theorem (1.6), whence that $\chi \in \mathrm{Irr}(G)$ is the unique irreducible character of G lying above the irreducible constituent λ of $\chi_{Z(G)}$ (see Theorem (1.11)). It is not difficult to see that $\chi(e) = 0$ if $e \in E \setminus Z(E)$ and that $\chi(z) = \chi(1)\lambda(z)$ if $z \in Z(E)$. The character χ is invariant in G, i.e. $\chi(geg^{-1}) = \chi(e)$ for each $g \in G$. We have $C_H(E) \trianglelefteq G$ and thus $EC_H(E) = E \times C_H(E)$. Therefore we can extend χ to an irreducible character $\bar{\chi} := \chi \otimes 1_{C_H(E)}$ of $EC_H(E)$. Note that $\ker \bar{\chi} = C_H(E)$ as χ is faithful. It follows that $\bar{\chi}$ is invariant in G. In the rest of the proof we work with $\bar{G} := G/C_H(E)$, $\bar{H} := H/C_H(E)$, $\bar{K} := KC_H(E)/C_H(E)$ and $\bar{E} := EC_H(E)/C_H(E)$ and $\bar{\chi}$. One easily checks that the groups $\bar{G}$, $\bar{H}$, $\bar{K}$, $\bar{E}$ satisfy the five hypotheses of the Theorem and that $\bar{\chi}$ is faithful on $\bar{E} \cong E$. As $C_H(E) \cap E = \{1\}$, we have $C_{\bar{H}}(\bar{E}) = \{\bar{1}\}$. So by an inductive argument we may also assume that $C_H(E) = \{1\}$. So G, H, K, E satisfy Hypothesis (2.6) with $C_H(E) = \{1\}$ and χ is faithful. Hence the result is immediate from Theorem (2.8). ♦

REMARKS. (1) Compare Hypothesis (2.1) with the hypotheses as given in Theorem (2.12).

(2) The proof of Theorem (2.11) used the contents of Theorem (2.8). Also the proof of Theorem (2.12) made use of the statements from Theorem (2.8). In the proof of Theorem (2.8) the contents of Theorem (2.5) were used. However, if $2 \nmid |E|$, then Theorem (2.5) was a direct corollary to Theorem (2.4). Dade (in [5]) remarked that for odd p a proof of Theorem (2.5) was described by Bolt, Room and Wall in 1961, see [3]. But if $2 \mid |E|$, then Theorem (2.5) was shown to be true also, by Dade in 1976 ([5], Theorem (1.2)). The proof for the $p = 2$-case was rather complicated and in 1978 Howlett [10] found a conceptual simplification of the proof of Theorem (2.5) by proving Theorem (2.12) in a comparatively easy and straightforward way under the supposition that K were solvable (which holds surely in the $p = 2$-case by the Feit-Thompson Theorem),

and that $C_H(E)=\{1\}$. Perhaps this is the place to observe that Howlett's ideas can also be applied on an arbitrary non-solvable K. Namely, Proposition 8 of [10] yields that we may assume that K is a minimal normal subgroup of H, so that K is the direct product of isomorphic copies of one non-abelian simple group X ([11], Satz I.9.12(a), (c)). Then Proposition 9 of [10] remains also valid because its proof requires the existence of a non-cyclic elementary abelian q-subgroup of K for at least one prime q. Its existence is trivial if $K \neq X$. So suppose $K=X$. Now K has a non-trivial Sylow 2-subgroup S, by the famous Feit-Thompson Theorem. So S contains a Klein four group, or it is cyclic, or it is (generalized) quaternion, as Satz III. 8.2 of [11] shows.

If S is cyclic, K has a normal 2-complement ([11], Satz IV.2.8) and so K is not simple. If K is (generalized) quaternion, then K is not simple either, by the Brauer-Suzuki Theorem ([11], Satz V.22.9). Anyway, for $q=2$ we have already that required existence of a non-cyclic elementary abelian q-subgroup of K, in order that Proposition 9 of [10] be valid. A further reading in Howlett's paper [10] reveals that K cannot be non-abelian simple unless Theorem (2.12) is true if $C_H(E)=\{1\}$. So a direct proof of Theorem (2.12) is possible along the lines of Howlett's. ♦

We now proceed to establish the one-to-one correspondences as mentioned in the introduction to this section.

(2.13) HYPOTHESIS. Let G, L and E be groups satisfying
(a) E is a normal subgroup of G.
(b) L is a normal subgroup of the factorgroup $G/C_G(E/Z(E))$.
(c) $\gcd(|L|, |E|)=1$.
(d) $[E/Z(E), L]=E/Z(E)$.
(e) $Z(E) \subseteq Z(G)$.
(f) $E/Z(E)$ is abelian.
(g) If $\tau \in \mathrm{Aut}(E)$ is trivial on $E/Z(E)$ and $Z(E)$, then $\tau \in \mathrm{Inn}(E)$.

REMARK. Notice that by Hypothesis (2.13)(e) $G/C_G(E/Z(E))$ can be embedded in $Sp(E)$. Also note that by Hypotheses (2.13)(c), (e), by Fitting's Lemma and by $E=C_E(L)[E,L]$ whenever $\gcd(|L|,|E|)=1$, Hypothesis (2.13)(d) is equivalent to $Z(E)=C_E(L)$.

The Hypothesis (2.13) leads to the existence of a "good" class of complements H to the section $E/Z(E)$ in G.

(2.14) THEOREM. *Assume* Hypothesis (2.13). *In the factorgroup $G_0=G/C_G(E)$ there is a unique conjugacy class of complements H_0 to the normal subgroup $E_0=EC_G(E)/C_G(E)$ of G_0. The inverse images of the H_0 form the unique conjugacy class of subgroups H of G satisfying*
(a) $G=HE$.
(b) $H \cap E=Z(E)$.
(c) $H \supseteq C_G(E)$.

PROOF. Hypothesis (2.13)(f) implies that $E_0 \simeq E/Z(E)$ is a normal abelian subgroup of G_0. By Hypothesis (2.13)(e) the group G_0 acts faithfully as automorphism group of E centralizing $Z(E)$. By Hypothesis (2.13)(g) it follows that the subgroup $C_G(E/Z(E))/C_G(E)$ of G_0 is precisely E_0 (note that $C_G(E/Z(E)) \supseteq EC_G(E)$). In particular there is a natural isomorphism

$$(1) \qquad G/C_G(E/Z(E)) \simeq G_0/E_0.$$

Now let L_0 be the normal subgroup of G_0 containing E_0 such that L_0/E_0 is the image of L under the isomorphism (1). Then Hypothesis (2.13)(c) tells us that $|L_0/E_0|$ and $|E_0|$ are coprime.

Since E_0 is a normal (abelian) subgroup of L_0, this implies by the Schur-Zassenhaus Theorem:

(2) *There is a unique conjugacy class of complements K_0 to E_0 in L_0.*

In view of Hypothesis (2.13)(d) and (1) any of these complements K_0 satisfies $[E_0, K_0] = E_0$. By Fitting's Lemma $E_0 = [E_0, K_0] \times C_{E_0}(K_0)$, so $C_{E_0}(K_0) = \{1\}$. Because $L_0 = K_0E_0$, $E_0 \trianglelefteq L_0$ and $K_0 \cap E_0 = \{1\}$ we have $N_{L_0}(K_0) = K_0 \cdot C_{E_0}(K_0) = K_0$. Now E_0 char $L_0 \trianglelefteq G_0$ so $E_0 \trianglelefteq G_0$. Let $g \in G_0$. Then

$$L_0^g = L_0 = (K_0E_0)^g = K_0^gE_0 \text{ and } K_0^g \cap E_0 = \{1\}.$$

So K_0^g is a complement to E_0 in L_0. So by the Schur-Zassenhaus Theorem there exists an $l = kx$ for some $k \in K_0$ and $x \in E_0$ such that $K_0^g = K_0^{kx} = K_0^x$. Hence $gx^{-1} \in N_{G_0}(K_0)$ and so $g \in E_0N_{G_0}(K_0)$. It follows that $E_0N_{G_0}(K_0) = G_0$. Moreover $E_0 \cap N_{G_0}(K_0) = N_{E_0}(K_0) = C_{E_0}(K_0) = \{1\}$. This and (2) imply that the normalizers $H_0^x = N_{G_0}(K_0^x)$ for $x \in E_0$ form the unique conjugacy class of complements to E_0 in G_0. Evidently their inverse images then form the unique conjugacy class of subgroups H of G satisfying (a), (b) and (c). This proves (2.14). ♦

REMARK. There is some resemblance with Theorem (1.4). See also Lemma (1.2)(b). However, Theorem (2.14) is independent of the order of E being even or odd. Moreover E does not have to be a $*$-group and we shall need the groups K_0, constructed in the proof of Theorem (2.14).

Now fix any of the subgroups H of Theorem (2.14). Using the conjugation action in G of H as automorphisms of E, we form the semidirect product of H and E, which we shall denote by $H \odot E$ to distinguish it from the product $G = HE$ of Theorem (2.14)(a). We shall identify E naturally with its image $\{1\} \odot E$ in $H \odot E$. This is legitimate since we do not so identify H (remember Theorem (2.14)(b)). Note that $|G| = |(H \odot E)/Z(E)|$, $G/E \simeq H/Z(E)$ and $(H \odot E)/E \simeq H$.

(2.15) PROPOSITION. *Assume* Hypothesis (2.13) *with E a $*$-group. Let $\chi \in \mathrm{Irr}(E)$ be faithful. Then χ is extendible to an irreducible character $\hat{\chi}$ of $H \odot E$, such that* $\ker(\chi) \supseteq C_G(E) \odot \{1\}$.

PROOF. Note that because E is a $*$-group Hypotheses (2.13)(f) and (2.13)(g) are automatically satisfied by Definition (1.1)(b) and Lemma (1.2)(b). In view of Theorem (2.14)(c) there is a natural surjective homomorphism of $H \odot E$ onto the semidirect product $H_0 \odot E$, where $H_0 = H/C_G(E)$. Since the kernel of this homomorphism is $C_G(E) \odot \{1\}$, we need only extend χ, considered as a character of $\{1\} \odot E \subseteq H_0 \odot E$, to an irreducible character of $H_0 \odot E$. The complement H_0 to E_0 in $G_0 = G/C_G(E)$ is naturally isomorphic to $G/C_G(E/Z(E))$ by (1) in the proof of Theorem (2.14). As in (2) in that proof let K_0 be the subgroup of H_0 corresponding to L. Then Hypothesis (2.13) implies that $H_0 \odot E$, $H_0 \odot \{1\}$, $K_0 \odot \{1\}$ and $\{1\} \odot E$ satisfy Hypothesis (2.9). So Theorem (2.11) gives the desired result. This proves (2.15). ♦

(2.16) THEOREM. *Assume* Hypothesis (2.13) *with E a $*$-group. Let λ be any faithful linear character of $Z(E)$ and χ the unique faithful irreducible character of E lying over λ by* Theorem (1.11). *If H is any of the subgroups of* Theorem (2.14) *and $\hat{\chi}$ is the extension of χ to the semidirect product $H \odot E$ of* Proposition (2.15), *then any character $\psi \in$* Irr $(H|\lambda)$ *determines a well defined character $\psi * \hat{\chi} \in$* Irr $(G|\chi)$ *by*

$$(\psi * \hat{\chi})(hx) = \psi(h)\hat{\chi}(h \odot x), \textit{ for all } h \in H \textit{ and } x \in E.$$

Furthermore, the map $F: \psi \longmapsto \psi * \hat{\chi}$ is a bijection from Irr $(H|\lambda)$ *to* Irr $(G|\chi)$.

PROOF. By Gallagher's Theorem ([13], (6.17)) the map $\psi \longmapsto \psi\hat{\chi}$ with $\psi\hat{\chi}(h \odot x) = \psi(h)\hat{\chi}(h \odot x)$ for all $h \in H$ and $x \in E$, sends Irr (H) one-to-one onto Irr $(H \odot E|\chi)$. In view of Theorem (2.14)(a)(b) the map $f: H \odot E \rightarrow HE$ defined by $f(h \odot x) = hx$ for all $h \in H$ and $x \in E$ is a surjective homomorphism with ker $(f) = \{z \odot z^{-1} \in H \odot E | z \in Z(E)\} \simeq Z(E)$. So composition with f maps Irr $(G|\chi)$ one-to-one onto the subset of all members of Irr $(H \odot E|\chi)$ having ker (f) in their kernels. Now $\chi|_{Z(E)} = \chi(1)\lambda$ and $\chi(1) = \sqrt{|E/Z(E)|}$. Now by Hypothesis (2.13)(e) and Theorem (2.14)(b) we have $Z(E) \subseteq Z(H)$. So if $\psi \in$ Irr (H), then $\psi|_{Z(E)} = \psi(1)\mu$ for some unique $\mu \in$ Irr $(Z(E))$. If $z \in Z(E)$ then we have

$$\begin{aligned}(\psi\hat{\chi})(z \odot z^{-1}) &= \psi(z)\hat{\chi}(z \odot z^{-1}) \\ &= \psi(z)\hat{\chi}(1 \odot z^{-1}) \quad (\text{because ker } (\hat{\chi}) \supseteq C_G(E) \odot \{1\}) \\ &= \psi(z)\chi(z^{-1}) \\ &= \psi(1)\chi(1)\mu(z)\lambda(z^{-1}).\end{aligned}$$

And this is equal to the degree $(\psi\hat{\chi})(1) = \psi(1)\chi(1)$ of $\psi\hat{\chi}$ if and only if $\lambda = \mu$. Hence ker $(\psi\hat{\chi}) \supseteq$ ker (f) if and only if $\psi \in$ Irr $(H|\lambda)$.

Also note that $(\psi * \hat{\chi})(1) = \psi(1)\sqrt{|E/Z(E)|}$. Further Irr $(G|\chi) =$ Irr $(G|\lambda)$ as χ and λ are fully ramified with respect to $E/Z(E)$. This proves (2.16). ♦

Now let G, H, L and E be groups satisfying Hypothesis (2.6). Now by Proposition (2.7)(d) it follows that the groups G, $L/C_H(E)$ and E satisfy Hypothesis

(2.13). For Proposition (2.7)(d) implies that $C_G(E/Z(E))=EC_G(E)$. And because by Hypothesis (2.6)(b) $G=HE$ is semidirect we have $C_G(E)=EC_H(E)$. Hence $C_G(E/Z(E))=EC_H(E)$. So $G/C_G(E/Z(E))=HE/EC_H(E)\simeq H/C_H(E)$. Hence Hypothesis (2.6)(c) implies Hypothesis (2.13)(b).

It follows that Theorem (2.14) applies. So there exists a unique conjugacy class of subgroups K of G satisfying

(a) $G=KE$.

(b) $K\cap E=Z(E)$.

(c) $K\supseteq C_G(E)$.

Now we have the following result analogous to Theorem (2.16).

(2.17) THEOREM. *Assume* Hypothesis (2.6). *Let* λ *be any faithful linear character of* $Z(E)$. *If* K *is any of the subgroups mentioned above, then there exists a one-to-one correspondence* F: Irr $(K|\lambda)\to$Irr $(G|\lambda)$ *such that for all* $\psi\in$ Irr $(K|\lambda)$, $F(\psi)(1)=\psi(1)\cdot\sqrt{|E/Z(E)|}$.

PROOF. Form the semidirect product $K\odot E$ as done just before Proposition (2.15). Then follow the proof of Theorem (2.16). This proves (2.17). ♦

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